Mu’tah University
Deanship of the Graduate Studies

The Cycles of Simple Graphs

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The Cycles of Simple graphs

استكمالاً لمتطلبات الحصول على درجة الماجستير في **الرياضيات.**

القسم: الرياضيات والإحصاء.

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Dedication

Present this work to my husband dear (Abu Hamza) who supported me to do this work.

Also my children dear.

To my mother and my father.

Also to my sister.

And all my family.

Alaa' Hakam Al-Rawajfeh

Acknowledgments

I thank Allah for helping me to complete this work.

I would like to express my sincere gratitude and thanks to my supervisor, Dr. Fawwaz Wrikat for his encouragement, continued to complete the work, generous support and valuable suggestion that without him would not appear in this final form.

I wish I could express my gratitude to every who contributed in the completion of this work.

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Alaa' Hakam Al-Rawajfeh

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Abstract

The Cycles of Simple Graphs

Alaa' Hakam Al-Rawajfeh

Mu'tah University, 2012

In this thesis, we study the cycle of some well known graphs.

We introduces new concepts and related to cycles of graphs and find the effect of operations on the cycles number and length.

We also introduce a new matrix representation of bipartite graphs, a new and original results on the relation of partition of bipartite Hamiltonian graphs onto two cycles.

Also the relation between the Characteristic polynomials, and the relation between girth and circumference.

ملخص

حول دارات البيانات

آلاء حكم الرواجفة

جامعة مؤتة، 2012

في هذه الرسالة، درسنا طبيعة الدارات لبعض أنواع البيانات المعروفة.

لقد قدمنا مصطلحات وتعاريف جديدة تخص أنواع معينة من البيانات، ودرسنا تأثرها بالعمليات العامة على البيانات من حيث العدد والطول لتلك الدارات ثم قمنا بتقديم تمثيل مصفوفي خاص وجديد لبيانات التجزئة وتمكنا من الحصول على نتائج جديدة تتعلق بالعلاقات بينها وبين البيانات الهملتونيه (Hamiltonian). وكذلك العلاقة بين المحيط وأصغر دارة وأثبتنا كيفية حسابهما ثم أوجدنا العلاقة بين كثير الحدود الممثل للبيانات وطبيعة الدارات المتعلقة بتلك البيانات.

Chapter One
Introduction and Basic Definitions
And Preliminaries of Graph Theory

Graph theory is an important field of mathematics which have a lot of applications in many sectors in science and engineering.

Graph theory is a branch of mathematics, it has a lot of applications in some areas of chemistry, physics, computer science, electrical engineering, genetics and economics, and their intimate relationships with many branches of mathematics including group theory, matrix theory, numerical analysis, probability and combinatorices, and that is important reason that lies behind the acceleration of interest in this subject. It must be admitted that the theory of graph spectra can be considered as a method of using matrix theory as a tool in graph theory and its applications; this makes the handling of graph theory easier as a theory of matrices is advanced. And simple on the other hand, it is possible to introduce the availability of high-speed computing machines to deal practically with graph theory and its applications. Consequently, the progress in graph theory, and its widespread usage in many fields of science, has taken place.

Nevertheless, this does not mean that the theory of graph spectra can be converteted purely into a simple or more general form of matrix theory. Still, the theory of graph spectra has not only its own nature but also its own applications, which reinforces its treatment as a theory in its own right.

In this thesis we will be mainly concerned with the investigation of retain relation of graphs and the characteristic polynomial of graphs and we will use matrix theory as a tool in graph theory and applications introduce some new results concerning the graph structure and linear algebra.

Chapter one is an introduction.

Chapter Two includes basic definitions and concepts of graph theory and preliminaries of graph theory that will be used in this thesis.

Chapter Three ,we exhibit a historical background and the work of some authors on the characteristic polynomial of graphs.

Chapter Four we will study the Bipartite graph that can be partitioned into two independent cycles.

Chapter Five is divided into three sections; In section one, we introduce a number of cycles, in section two, definition and new theorem about girth and circumference. In section three, we investigate the relation between the cycles matrices and the characteristics polynomial.

Chapter Two

Basic Definitions and Preliminaries

In this chapter, we present some basic definitions and preliminaries of graph theory which are needed later in this thesis. For the material of this chapter see (Harary. F, 1972).

Definition 2.1. A graph Γ is an unordered triple $(V(\Gamma),E(\Gamma),\Psi_{\Gamma})$ consisting of a non-empty set $V(\Gamma)$ of Vertices (points), a set $E(\Gamma)$ of edges (lines) and an incidences function:

$$Y_G : E(G) \longrightarrow \left\{ \binom{V(G)}{2} \right\}$$

associates with each edge of G an ordered pair of vertices of G where $\binom{V(G)}{2}$ is the set of unordered vertices.

Note 2.1 we write $x=(u,v)$ and say that u and v are adjacent vertices; vertex u and edge are incident with each other, as are v and x.

Definition 2.2. Let Γ be a given graph. Then the number of vertices in Γ is called the ordered of Γ and is denoted by $|V(\Gamma)|$. The number of edges of Γ is called the size of Γ and is denoted by $|E(\Gamma)|$.

Definition 2.3. If v_i is adjacent to itself, we call it a loop. If there exists at least two edges incident with a vertex v_i and a vertex v_j, we call these edges parallel (multiple) edges .

Definition 2.4. A graph Γ is undirected graph. That is to say if $(u,v)\in E(G)$, then $(v,u)\in E(G)$.

Definition 2.5. A graph Γ is called a simple graph if it is undirected and it has no loops and no parallel edges .

Definition 2.6. The valency (degree) of a vertex v in $V(\Gamma)$, denoted by *val(v)*, is the number of edges incident with *v*.

Definition 2.7. A graph Γ is called *k*- regular if *val(v)=k* for all $v \in V(\Gamma)$.

Definition 2.8. A subgraph $\Phi = (V(F), E(F))$ of a graph $\Gamma = (V(\Gamma), E(\Gamma))$ is a graph with $V(\Phi) \subseteq V(\Gamma)$ and $E(\Phi) \subseteq E(\Gamma)$.

Definition 2.9. A walk in a graph Γ is an alternating sequence of vertices and edges $\{v_0, x_1, v_1, \ldots, v_{n-1}, x_n, v_n\}$ beginning and ending with vertices.

Definition 2.10. A trial is a walk with distinct edges.

Definition 2.11. A path is a walk with distinct edges and distinct vertices. Its length is the number of its edges.

Definition 2.12. A cycle (circuit) of length *n* is a closed path denoted by C_n.

Definition 2.13. The girth of a graph Γ is the length of the shortest cycle in Γ, denoted by $g(G)$, and the circumference is the longest cycle denoted by $C(G)$.

Definition 2.14. A complete graph K_n is a graph with $|V(K_n)| = n$, $|E(K_n)| = \frac{1}{2}(n^2 - n)$, and every vertex is adjacent with all the other vertices. See fig. 2.1.

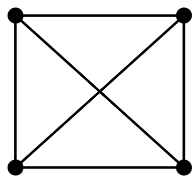


Fig 2.1 A Complete Graph

Definition 2.15. A complete bigraph $K_{m,n}$ is a graph with two disjoint sets of vertices I, II where $|V(I)| = m, |V(II) = n|$, and every vertex

in I adjacent with all vertices in II and the vertices in the same set are not adjacent and $|E(K_{m,n})| = mn$, $|V(K_{m,n})| = m + n$. See fig 2.2.

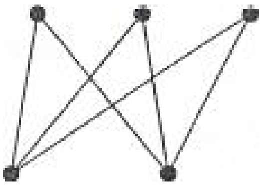
$K_{3,2}$:


Fig 2.2 A Complete Bigraph

Definition 2.16. A graph Γ is connected if there exists at least one path between any two distinct vertices in Γ.

Definition 2.17. The vertex connectivity of the graph Γ is the minimum number of vertices whose removal cause a disconnected or trivial graph and is denote by $VC(\Gamma)$.

Definition 2.18. The edge connectivity at set of Γ denoted by $EC(\Gamma)$ is the minimum number edges whose removal cause a disconnected or trivial graph.

Definition 2.19. Let Γ_1 and Γ_2 be two graphs. Then Γ_1 union Γ_2 denoted by $G_1 \cup G_2$, is a graph with $V(\Gamma_1 \cup \Gamma_2) = V(\Gamma_1) \cup V(\Gamma_2)$ and $E(\Gamma_1 \dot{\cup} \Gamma_2) = E(\Gamma_1) \cup E(\Gamma_2)$.

Definition 2.20. Let Γ_1 and Γ_2 be two graphs. Then Γ_1 join Γ_2 denoted by $G_1 \dot{+} G_2$ is a graph with $V(G_1 \dot{+} G_2) = V(G_1) \cup V(G_2)$, $E(G_1 \dot{+} G_2) = E(G_1) \cup E(G_2)$ and all edge joining $V(G_1)$ with $V(G_2)$.

Example 2.1. If

G_1:


G_2:

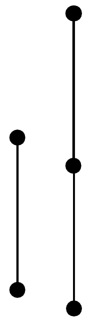
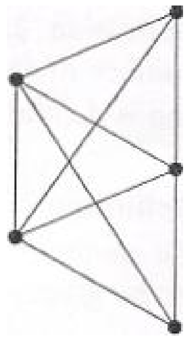

Then $G_1 \cup G_2$  , $G_1 \dot{+} G_2$: 

Fig 2.3 The Graphs

Definition 2.21 The wheel $W(I,n)$ is defined to be the $K_1 + C_n$. (See Fig.2.4).

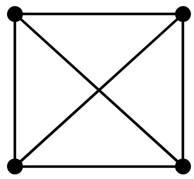
$W(1,4) = K_1 + C_4$: 

Fig.2.4. A wheel

Definition 2.22. The complement $\overline{G}$ of a graph G has $V(G)$ as its vertex set, but two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G, and non adjacent if they are adjacent in G.

Note 2.2. Let G be a simple graph. Then $G \cup \overline{G} = K_n$; where $|V(G)| = n$.

Definition 2.23. A cut vertex of graph is a vertex whose removal disconnect the graph and a bridge is such an edge (line) .

Definition 2.24. A graph is called nonseprable if it has no cut vertex.

Definition 2.25. The line graph of a graph Γ, denoted by L(Γ), the graph such that two vertices of L(Γ) are adjacent when ever the corresponding edges of Γ are incident.

Example 2.4 If

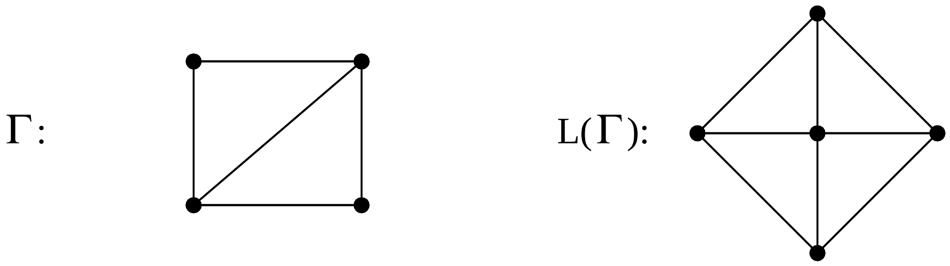


Fig 2.6. Graph and their Line Graphs

Definition 2.26. A graph Γ is labeled when p vertices are distinguished from one another by names such as $v_1, v_2, \ldots, v_p$, for every integer $P \leq |E(G)|$.

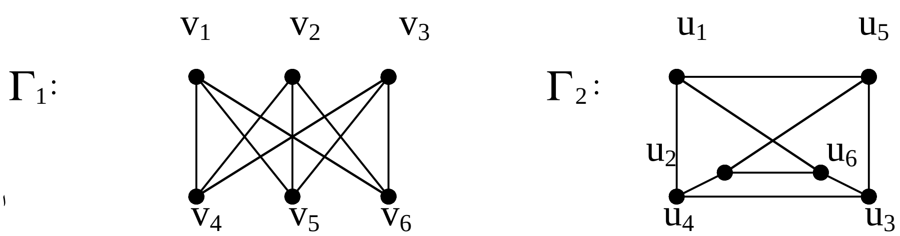
Definition 2.27. A connected graph Γ is Hamiltonian if there is a cycle which includes every vertex of Γ (spanning cycle); such a cycle is called a Hamiltonian cycle. If any path of Γ includes every vertex, then it is called a Hamiltonian path.

Definition 2.28. A theta graph is a block with two non adjacent vertices of degree 3 and all other vertices of degree 2.

Thus a theta graph contains two vertices of degree 3 and three disjoint paths joining them each of length at least 2.

Definition 2.29. Two graphs Γ_1 and Γ_2 are said to be isomorphic denoted by $\Gamma_1 \cong \Gamma_2$, if there exists one-to-one correspondence between their vertex sets which preserves adjacency .

Example 2.6.



Γ_3:

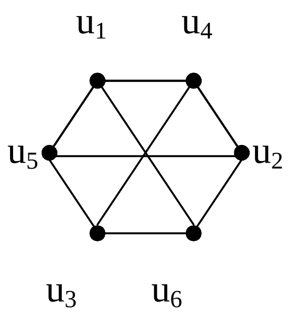


Fig 2.7. Three isomorphic graphs

Definition 2.30. The distance $d(u,v)$ between two vertices u and v in Γ is the length of the shortest path joining them if any, otherwise $d(u.v) = \infty$.

In a connected graph. The distance is metric that is for all vertices u, v and w,

1. $d(u,v) \geq 0$ with $d(u,v) = 0$ if and only if $u = v$.
2. $d(u,v) = d(v,u)$.
3. $d(u,v) + d(v,w) \geq d(u,w)$.

Definition 2.31. The adjacency matrix $A(\Gamma) = [a_{ij}]$ of a labeled graph Γ with n vertices $\{v_1, v_2, \mathbf{K}, v_n\}$ is an $n \times n$ matrix in which

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise} \end{cases}.$$

Thus there is one-to-one correspondence between labeled (simple) graphs with n vertices and $n \times n$ symmetric binary matrices with zeros diagonal .

Definition 2.32. The determinant of matrix ($l\mathrm{I} - \mathrm{A}$) is called the characteristic polynomial of A. The characteristic polynomial $|l\mathrm{I} - \mathrm{A}|$ of the adjacency matrix $A(\Gamma) = A$, is called the characteristic polynomial of the graph Γ and denoted by $P_\Gamma(l) = a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_n$.

Chapter Three

Review of the Earlier Studies

In this chapter, we exhibit the earlier studies on the cycles of graphs.

3.1 On the maximum number of cycle in a Hamiltonian graphs:

First, we give a list of notation, which will be used in this section.

Let $M(t)$ denote the maximum number of cycles in Hamiltonian graph of order $n=|V(G)|$ and size $n+t=|E(G)|$.

If $f(G)$ denotes the number of cycles of graph G, then $M(t)=Max\{f(\Gamma)|\Gamma\in\Gamma_t\}$ where G_t denotes the set of all simple graph.

We will assume that for each graph G in G_t is a one Hamiltonian cycle $C_G: u_1,u_2,...,u_n$ has been fixed and we call the additional edges in $E(G)/E(C_G)$ chords. Tow chords u_1v_1 and u_2v_2 of some $G\in G_t$ are called skew, if their end vertices appear in the cycle order u_1,u_2,v_1,v_2 on C_G. Two chords are parallel, if they are not skew.

Theorem 3.1.1.[Y. Shi 1994]. For $t\geq 1$

$$2^t+t(t-1)+1\leq M(t)\leq 2^{t+1}-1 \quad \ldots \qquad (1)$$

and for $1\leq t\leq 4$ equality holds in left side of (1). The next lemma describes some properties of extremal graphs.

Lemma 3.1.1. [Dieter Rautenbach and Irene Stella, 2005]

1. If $\Gamma\in\Gamma_t$ and $f(\Gamma)=M(t)$, then no two chords are incident with the same vertex, i.e. the maximal degree $\Delta(\Gamma)$ of Γ is at most 3.
2. If $\Gamma\in\Gamma_t$ with $f(\Gamma)=M(t)$, then there is no set of chords $S\subseteq E(\Gamma)/E(C_\Gamma)$ such that $\phi\neq S\neq(E(\Gamma)/E(C_\Gamma))$ and each chord in S is parallel to each chord not in S.
3. For $t\geq 2$, there is a 3-regular graph $\Gamma\in\Gamma_t$ with $f(\Gamma)=M(t)$.

4. For $t \geq 2$, every graph $H \in \Gamma_t$ with $f(H) = M(t)$ arises from a 3-regular graph $\Gamma \in \Gamma_t$ with $f(\Gamma) = M(t)$ by replacing the edges of C_Γ by non-trivial paths.

Lemma 3.1.2. [Dieter Rautenbach and Irene Stella 2005]

1. If $\Gamma \in \Gamma_t$ is such that $C_\Gamma : u_t u_2 \ldots u_1 v_1 v_2 \ldots v_t$ and $E(G)/E(C_G) = \{u_i v_i | 1 \leq i \leq t\}$, then $f(G) = 1 + 2^t + 2\binom{t}{2}$.
2. If $\Gamma \in \Gamma_t$ is such that $C_\Gamma : u_t u_2 \ldots u_1 w v_1 v_2 \ldots v_t w$ and $E(G)/E(C_G) = \{u_i v_i | 1 \leq i \leq t\}$, then $f(G) = 1 + 2 + \binom{t}{2}$.

Lemma 3.13. [Dieter Rautenbach and Irene Stella 2005] Let $G = (V(G), E(G))$ be a graph of order n and let $r = \lfloor log_2(\sqrt{n}) + 1 \rfloor$. There are $\lfloor n/2r \rfloor + 1$ disjoint subsets of $V(G)$ of size r that each of the set induces either a clique or an independent set.

Theorem 3.1.1. [Dieter Rautenbach and Irene Stella 2005] For $k \geq 5$

$$M(t) \geq 2^t + \frac{5}{2}t^2 - \frac{21}{2}t + 14 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (2)$$

And

$$M(t) \leq 2^{t+1} - 1 + t\left(\frac{\sqrt{t} - 2}{log_2(t) + 2} - \frac{1}{4}log_2(t)\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (3)$$

3.1.1 Results on *M(t)* for $5 \leq t \leq 10$:

In this section we report computational results regarding $M(t)$ for $5 \leq t \leq 10$. In order to determine $M(t)$ and the structure of the extremal graphs for fixed t, Lemma 1 (4) implies that we can restrict our considerations to Hamiltonian, 3-regular graphs of order 2t having 2t edges.

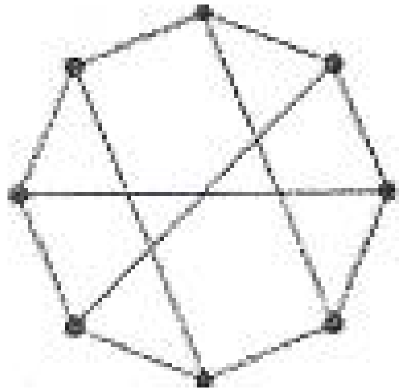


Fig.3.1.1. Extremal graph for t=4 with 29 cycles

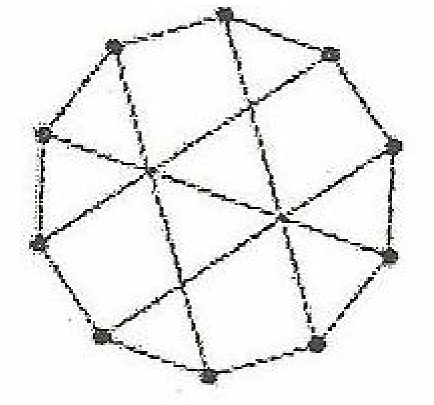


Fig.3.1.2. Extremal graph for t=5 with 56 cycles

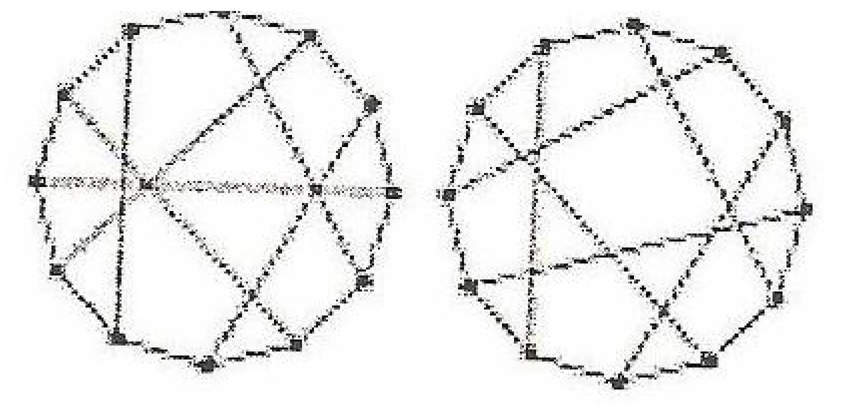


Fig.3.1.3. Extremal graph for t=6 with 109 cycles

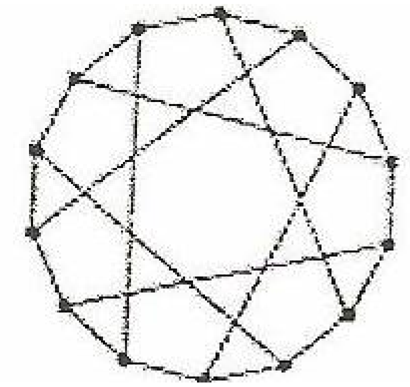


Fig.3.1.4. Extremal graph for t=7 with 213 cycles

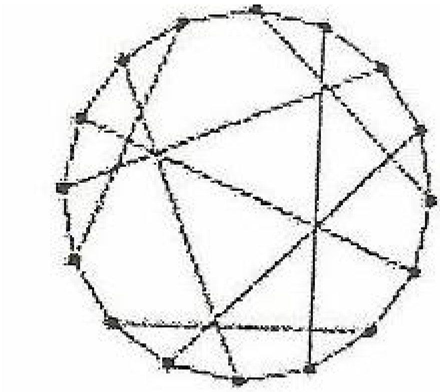


Fig.3.1.5. Extremal graph for t=8 with 401 cycles

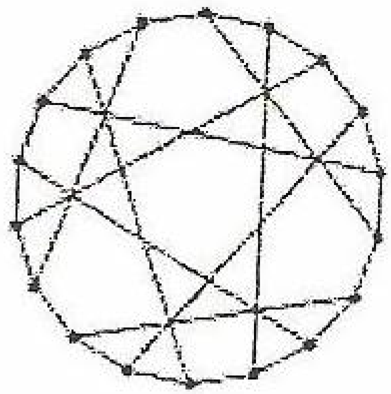


Fig.3.1.6. Extremal graph for t=9 with 783 cycles

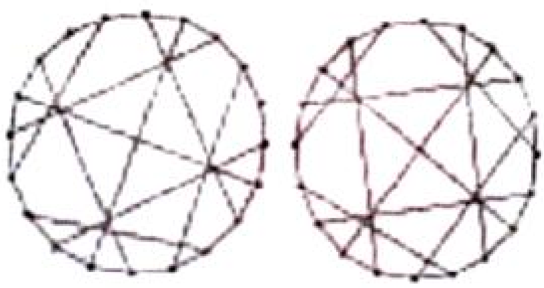


Fig.3.1.7. Extremal graph for t=10 with 1484 cycles

3.2. The Construction of (k, g-s)-cages.

In this section we give results related to (k,g-s) for an even positive integer k for s =1, 2, ..., g-3; where $g \geq 6$.

Notation 3.2.1. For positive integers $k \geq 2$ and $g \geq 3$ a [k, g] -graph is a graph having minimum order. The order of a [k, g]-graph is denoted by $n(k$, g). The [k, g] -graph are commonly referred to as [(k, g)] -cages. The Tutte's lower bound of [k, g]-graph denoted by $n_0(k, g)$, where

$$n_0(k,g) = \begin{cases} k(k-2)^{-1}\left[(k-1)^t - 1\right] + 1 & if \;\; g = 2t+1, \\ 2k(k-2)^{-1}\left[(k-1)^t - 1\right] & if \;\; g = 2t \end{cases}$$

Let A be a subgraph of a graph Γ and let $v \in V(\Gamma)$. The distance $d(A, v)$ is defined as follows:

$$d(A,v) = \begin{cases} min(d(v,u)) & if \;\; v \notin V(A), u \in V(A), \\ 0 & if \;\; v \in V(A). \end{cases}$$

We define $R_j(A), S_j(A)$ and $T_j(A)$: where $j \in Z^+$ as follows:

$$R_j(A) = \{y \in V(\Gamma) \quad : \quad d(A,y) = j\},$$
$$S_j(A) = \{y \in V(\Gamma) \quad : \quad d(A,y) \leq j\},$$

and let T_j (A) is the subgraph of Γ whose vertices of S_j (A) and whose edges are the edges of Γ incident with at least one vertex of S_{j-1}. Let $T_j(A)$ be the subgraph of Γ whose edges are $E(\Gamma) - E(T_j(A))$ and whose vertices are the members of $V(\Gamma) - V(T_j(A))$ together with vertices of attachment of $T_j(A)$ in Γ.

Theorem 3.2.1. [F. Doujan Wrikat 2001]. Let k be an even positive integer and $g \geq 6$. Then for $s = 1, 2, \ldots, g-3$,

$$n(k, g-s) \leq n(k,g) - n_0(k,s)$$

3.3 Partition of graph into two cycles:

Theorem 3.3.1. [Amar 1986]. Let Γ is be graph with vertices sets A and B such that $A \cap B = \phi, |A| = |B| = n$ and for any $x \in A$ and $y \in B$, $d(x) + d(y) \geq n + 2$, then for any $(n_1, n_2), n_1, n_2 \geq 2$ $n = n_1 + n_2, \Gamma$ contains two independent cycles of length $2n_1$, and $2n_2$.

Example 3.3.1. Let Γ be a bipartite graph such that, $|V(\Gamma)| = 10$, $A = \{x_1, \ldots, x_5\}, B = \{y_1, \ldots, y_5\}$ and the condition $d(x_i) + d(y_i) \geq 7$ for all $x_i \in A, y_j \in B, i, j \in \{1, \ldots, 5\}$, see fig 3.4.1. By theorem 3.4.1, Γ contains two independent cycles of length 6, 4;

$$C_1 = x_1 y_1 \, x_2 y_2 \, x_3 y_3 \, x_1$$
$$C_2 = x_4 y_4 \, x_5 y_5 \, x_4$$

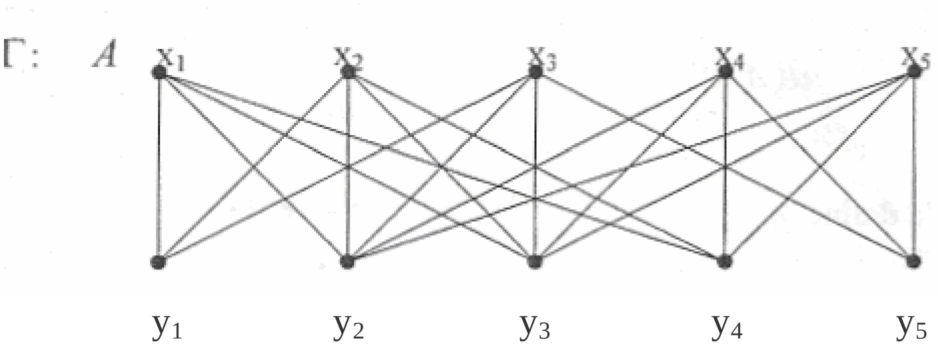


Fig 3.3.1. A graph Γ

Lemma 3.3.1. [Amar 1986]. If for any $x \in A, y \in B$, $d(x, \Gamma) + d(y, \Gamma) \geq n + 1, \Gamma$ is Hamiltonian.

Lemma 3.3.2. [Amar 1986]. If Γ contains a Hamiltonian path with end vertices a and b such that $d(a, \Gamma) + d(b, \Gamma) \geq n + 1$ then Γ is Hamiltonian.

Theorem 3.3.2. [AL-Najjar 2001]. Let $\Gamma = (V_1, V_2; E)$. Suppose $|V_1| = |V_2| = n = n_1 + n_2$ with $n_1 \geq n_2 \geq 2$ and $\delta(\Gamma) \geq n_1 + (n_2)/2$. Then Γ contains two vertex-disjoint cycles of lengths $2n_1$, and $2n_2$, respectively.

Example 3.3.2. Let $\Gamma = (X, Y; E)$ be such that $X = \{x_1, ..., x_5\}$, $Y = (y_1, ..., y_5)$,

$E = \{x_1y_1, x_2y_2, x_3y_3\} \cup \{x_i, y_j : i = 4,5 \, and \;\; j = 1, 2, 3\} \cup \{x_i y_j \; : i = 1,2,3 \, and \;\; j = 4,5\}$

Then Γ contains a cycle of length 6 and a cycle of length 6 such as; $C_1 = x_4y_1x_1y_3x_5y_2x_4$, $C_2 = x_1y_4x_2y_5x_1$. But the graph does not contain two vertex-disjoint cycles of length 4, 6 since

$$\delta(\Gamma) = 3 > n_1 + (n_2)/2 = 4.$$

Theorem 3.3.3. [AL-Najjar 2001]. Let $\Gamma(V_1, V_2; E)$. Suppose $|V_1| = |V_2| = n = n_1 + n_2 + ... + n_k$ with $n_1 \geq n_2 \geq ... \geq n_k \geq 2$ and $\delta(\Gamma) \geq n_1 + n_2 + ... + n_{k-1} + (n_k)/2$. Then Γ contains k vertex-disjoint cycles of lengths $2n_1, ..., 2n_k$, respectively.

Example 3.3.3. [AL-Najjar 2001].

Let $\Gamma = (X, Y; E)$ be such that; $X=\{x_1,...,x_6\}$, $Y=\{y_1,...,y_6\}$, E=$\{x_1y_2, x_3y_2, x_3y_4, y_4x_1, x_2y_3, x_4y_3, x_4y_1, x_2y_1, x_6y_5, x_5y_5, x_5y_6, x_6y_6\}$, $\delta(\Gamma) = 2 < n_1 + n_2 + (n_3)/2 = 2 + 2 + 1 = 5, n_1 = n_2 = n_3 = 2.$

Γ contains three vertex-disjoint cycles of lengths $2n_1$, $2n_2$, $2n_3$ i.e. of length 4 $C_1 = x_1y_2x_3y_4x_1$, $C_2 = x_2y_3x_4y_1x_2$ and $C_3 = x_6y_5x_5y_6x_6$ are three vertex disjoint cycles of length 4.

Theorem 3.3.4. [AL-Najjar 2001]. Let $D = (V_1, V_2;\; A)$ be a directed bipartite graph with $|V_1| = |V_2| = n \geq 2$. Suppose $d_D(x) + d_D(y) \geq 3n + 1$, for all $x \in V_1$ and $y \in V_2$. Then D contains two vertex disjoint directed cycles of lengths $2n_1$, and $2n_2$ respectively, for any positive integer partition $n = n_1 + n_2$.

3.4. Bipartite graph containing every possible pair of cycles:

Theorem 3.4.1. [Wang 1999]. Let $\Gamma = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = |V_2| = n \geq 4$. Suppose $d(x) + d(y) \geq n + 2$, for all $x \in V_1$

and $y \in V_2$. Then Γ contains two vertex disjoint cycles of lengths 2*s* and 2*t* respectively, for any positive integers *s* and *t* with $s \geq 2, t \geq 2$ and $s + t \leq n$.

Note3.4.1. [Wang 1999]. In the following $\Gamma = (V_1, V_2; E)$ is a bipartite graph $|V_1| = |V_2| = n \geq 2$.

Lemma 3.4.2. [Wang 1999]. Let $p = x_1 y_2 ... x_k y_k$ be a path of Γ. Let $y \in V(\Gamma)$ be a vertex not on *p* such that $\{x_1, y\} \not\subset V_i$ for each $i \in \{1,2\}$, if that $d(x_1, p) + d(y, p) \geq k + 1$, then Γ has a path p' from *y* to y_k such that $V(P') = V(P) \cup \{y\}$.

Lemma 3.4.3. [Wang 1999]. If $d(x) + d(y) \geq n + 2$ for pair of non-adjacent vertex $x \in V_1$ and $y \in V_2$. Then the following two statement holds:

(a) If $n \geq 3$, then $\Gamma - x - y$ is Hamiltonian for all $x \in V_1$ and $y \in V_2$.

(b) For any three distinct vertices $x \in V_1, y_1 \in V_2$ *and* $y_2 \in V_2; \Gamma - x$ has a Hamiltonian path from y_1 to y_2.

Lemma 3.4.4. [Wang 1999]. If $d(x) + d(y) \geq n + 1$ for each $x \in V_1$ and $y \in V_2$, then Γ contains a cycle of length 2*k* for each $k \in \{2, 3, ..., n\}$, unless *n* = 3 and Γ is a cycle of length 6.

Lemma 3.4.5. [Wang 1999]. Let *C* be a cycle of length 2*k* in Γ. Let *xy* be an edge in $\Gamma - V(C)$ such that $d(x, C) + d(y, C) \geq k$. If *C* + *x* + *y* is not Hamiltonian then either $d(x, C) = 0$ or $d(y, C) = 0$.

Lemma 3.4.6. [Wang 1999]. Let *p, q* and *k* be three positive integer with $k \geq P$ and $k+q<n$. Let *H* be an induced traceable subgraph of order *2k* in Γ. Let *F* be traceable of order *2q* in Γ-*V*(*H*). Let z_1 and z_2 be two vertices from $\Gamma - V(H \cup F)$ with $z_1 \in V_1$ and $z_2 \in V_2$. Set $D = \Gamma - V(H \cup F) - \{z_1, z_2\}$. Then the following hold.

(a) $d(u, H) + d(v, H) \geq k + 2$ for a1l $u \in V_1 \cap V(H)$ and $v \in V_2 \cap V(H)$.

(b) $d(x,F)+d(y,F)\geq q+2$ for a1l $x\in V_1\cap V(F)$ and $y\in V_2\cap V(F)$.

(c) $d(z_1,H)+d(z_2,H)\geq k+2$ and $d(z_1,F)+d(z_2,F)\geq 1$.

3.5. Spectra of cycles .

Theorem 3.5:1. [Cvetkovic 1978]. Let Γ be a regular graph with girth $g(\Gamma)=g$ and with the characteristic polynomial $P_\Gamma(\lambda)=\lambda^n+a_1\lambda^{n-1}+a_2\lambda^{n-2}+...+a_n$. Let $h\leq n$ be a nonnegative integer not greater than 2*g*-1 and not less than 3. Then the number of circuits of length *h*, which are contained in Γ, is determined by the largest root *r* and the first *h* coefficients of $P_\Gamma(\lambda)$.

Definition 3.5.1. An $n\times n$ matrix *S* is said to be a circulant matrix if its entries satisfy $s_{ij}=s_{1,j-i+1}$ where the subscripts are reduced module *n* and lie in the set *1, 2, ..., n.* In other words, row *i* of *S* is obtained from the first row of *S* by a cycle shift of *i* -1 steps. So any circulant matrix is determent by its first row. Let *W* denote the circulant matrix whose first row is [0,1,0,...,0].i.e.

$$W=\begin{bmatrix}0&1&0&0&...&0\\0&0&1&0&...&0\\0&0&0&1&...&0\\\vdots&\vdots&\vdots&\vdots&...&\vdots\\1&0&0&0&...&0\end{bmatrix}$$

And let *S* denote a general circulant matrix whose first row $[s_1,s_2,...,s_n]$ then $S=\sum_{j=1}^{n}s_jW^{j-1}$.

Remark 3.5.1. [Biggs 1974]. Since the eigenvalues of *W* are $1,w,w^2,...,w^{n-1}$, where $w=exp(2\pi i/n)$ it follows that the eigenvalues of *S* are $\lambda_r=\sum_{j=1}^{n}S_jw^{(j-1)(r-1)},r=1,2,...,n$.

Note 3.5.1. A circulant graph whose adjacency matrix $A(\Gamma)$ is a circulant matrix.

Proposition 3.5.1. [Wrikat 1992]. Suppose [0, a_2, ..., a_n] is the first row of adjacency matrix of a circulant graph Γ.

Then the eigenvalues of Γ are $\lambda_r = \sum_{j=2}^{n} a_j w^{(j-1)(r-1)}, r = 1,2,\ldots,n$.

Remark 3.5.2. [Biggs 1974]. The spectrum of C_n is

$$S_P(C_n) = \begin{cases} 2\left(2cos\frac{2\pi}{n}\right)^2 \ldots \left(2cos\frac{(n-1)\pi}{n}\right)^2 & \textit{if n is odd}, \\ 2\left(2cos\frac{\pi}{n}\right)^2 \ldots \left(2cos\frac{(n-2)\pi}{n}\right)^2 - 2 & \textit{if n is even}, \end{cases}$$

Proposition 3.5.2. [Biggs 1974]. Let Γ be a *k*- regular graph, let $P_\Gamma(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_n$ be the characteristic polynomial of the adjacency matrix A of Γ. Then the following hold:

(a) $a_1 = 0$.

(b) $a_2 = -|E(\Gamma)|$.

(c) $-a_3/2$ = the number of 3-cycles in Γ.

Definition 3.5.2. A graph in which every' component is isomorphic to K_2 or C_q is called a basic figure, K_2, C_q are called elementary figures.

Theorem 3.5.2. [Cvetkovic 1978]. Let Γ be a graph and let $P_\Gamma(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_n$ be the characteristic polynomial of the adjacency matrix *A* of Γ. Then the length f of shortest odd circuit in Γ equal to the index of the first non-vanishing coefficient among $a_1, a_3, a_5, \ldots$ The number of shortest odd circuits is equal to $\frac{-1}{2}a_f$.

Theorem 3.5.3. [Cvetkovic 1978]. Let Γ be a graph with $P_\Gamma(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_n$ and let b_q be the basic figures. Let

$$\tilde{a}_i = \begin{cases} a_i & \text{for odd } i \\ a_i - (-1)^q b_q & \text{for } i = 2q \end{cases}$$

Then the girth g of Γ equals the index of the first non-vanishing coefficient $\tilde{a}_i$ among $\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n$, and the number of circuits is equal to $\frac{-1}{2}\tilde{a}_i$.

Chapter Four

Partition of a Bipartite Hamiltionian Graph Into Two Cycles

4.1 Introduction:

We discuss only finite, simple graphs. J.A. Bondy and V.Chv' atal proved that if G is a bipartite graph with bipartition (A, B) such that $|A|=|B|=n$ and if for any $x\in A, y\in B, d(x)+d(y)\geq n+1$, then G is Hamiltionian.

A bipartite graph with bipartition (A, B) is called balanced if $|A|=|B|$.

In (3) our aim is to prove the following theorem.

Theorem. If G is a balanced bipartite graph with bipartition (A, B), $|A|=|B|=n$ such that for any $x\in A, y\in B, d(x)+d(y)\geq n+2$, then for any $(n_1, n_2), n_i\geq 2, n=n_1+n_2$, G contains two independents cycles of lengths $2n_1$ and $2n_2$.

Obviously that the above condition in the previous theorem gives that graph G is Hamiltionian.

This material is essentially in (Amar, 1986).

The main theorem in this situation is introduced in the following.

Main Theorem

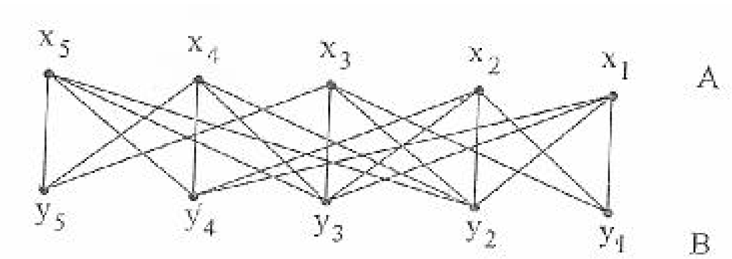
If G is a balanced bipartite graph with bipartition (A, B), $|A|=|B|=n$ such that for any $x\in A, y\in B, d(x)+d(y)\geq n+2$, then for any (n_1, n_2), $n_i\geq 2, n=n_1+n_2$, G contains two independent cycles of lengths $2n_1$ and $2n_2$.

1- Example:

Let G be a bipartite graph such that, $|V(G)|=10$, $A=\{x_1,\ldots,x_5\}$, $B=\{y_1,\ldots,y_5\}$ and the condition $d(x_i)+d(y_j)\geq 7$ for all

$x_i \in A, y_j \in B, i, j \in \{1,\ldots,5\}$, see fig.5.3.1 By Theorem, G contains two independent cycles of lengths 4, 6;

$$C_1 = x_1 y_1 x_2 y_2 x_3 y_3 x_1$$
$$C_2 = x_4 y_4 x_5 y_5 x_4$$



sceam. 1

Note that, the condition in the Theorem 4.1.1 is sharp.

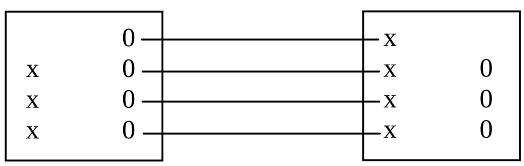
Notation:

If *F* and *H* are disjoint subgraphs of *G* and if $u_1,\ldots,u_p$ are vertices of *G*, not in *H*, then $H + (u_1,\ldots,u_P)$ is the subgraph of *G* with vertices $u_1,\ldots,u_P$ and the vertices of *H*.

If v_l, ..., v_k are vertices of *H*, *H*-(v_l,...,v_k) is the subgraph of *G* with vertices in *H* except v_l, ..., v_k. If *C* is a path or a cycle of *G*, then for any arbitrary orientation; if *u* is a vertex of *C*, u^+ (resp. u^-) is the successor (resp. the predecessor) of u on the path or the cycle for the given orientation.

Remark:

If *n* is odd, the result of the theorem is the best one as we can see with the following example: *n*=2*p*+1. There is no partition into two cycles of lengths 2*p* and 2*p*+2 (see e.g. sceam.2).



sceam. 2

It is clear that every vertex in G in sceam.2 of degree 4, so $d(x)+ d(y)=8$ and $|V(G)|=14$ but G does not contain two independent cycles of lengths 6, 8.

The proof of Theorem is based on many elementary lemmas that we give first.

4.2 Elementary Lemmas:

Let G be a balanced bipartite graph with bipartition (A, B), such that $|A|=|B|=n$.

Lemma:

If for any $x\in A, y\in B, d(x,G)+d(y,G)\geq n+1$, G is Hamiltionian.

Proof. Assume G is not Hamiltionian, with $d(x,G)+d(y,G)\geq n+1$ for each two vertices $x\in A, y\in B$. We may assume by adding more edges to G if necessary, that G is only just non-Hamiltionian, in the- sense that the addition of any more edges would make it Hamiltionian, then any two nonadjacent vertices must be joined by a spanning path.

Let $v_1v_2v_3...v_{2n}$ be a Hamiltionian path of G and $v_1v_{2n}\notin E$. Since $v_1\in A, v_{2n}\in B$, then we must have $d(v_1,G)+d(v_{2n},G)\geq n+1$, i.e. $d(v_{2n},G)\geq n-d(v_1,G)+1$.

It follows if $d(v_1,G)=r$, then there are at most r-1 vertices non-adjacent to v_2 since $d(v_{2n},G)\geq n-r+1$.

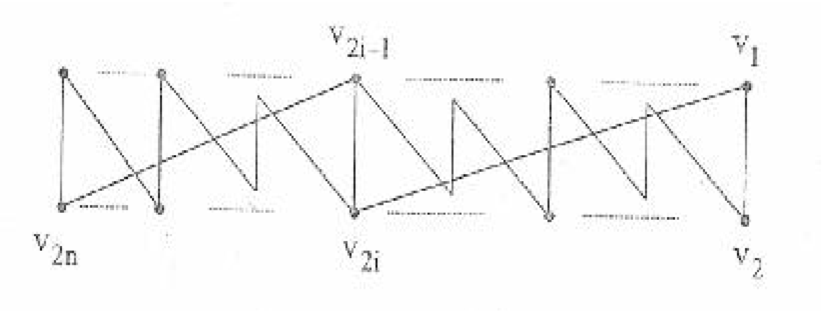
Now, consider the vertices adjacent to v_1, and let S be the set of the vertices preceding each of these vertices in the -path, i.e. if $v_1v_{2k}\in E$ then $v_{2k-1}\in S$, then S contains r vertices and $v_{2n}\notin S$.

It follows from the two italicized statements that S must contain a vertex $v_{2i\text{-}1}$ adjacent to v_{2n}, and so there must be edges

joining v_1 and v_{2i}, as shown in the below diagram. But we can now write down a Hamiltionian cycle in G-namely,

$$v_1 v_2 \ldots v_{2i-1} v_{2n} v_{2n-1} \ldots v_{2i} v_1 \text{ (see Fig.3)}$$

Contradicting the assumption that *G* is non-Hamiltionian. This contradiction establishes the lemma.



sceam. 3

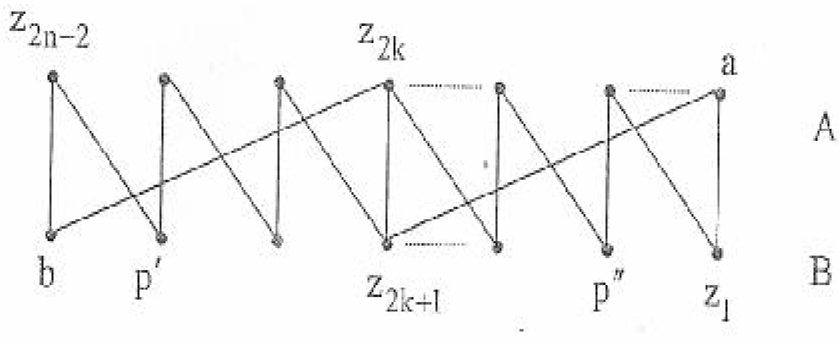
Lemma:

If *G* contains a Hamiltionian path with endvertices *a* and *b* such that $d(a,G)+d(b,G)\geq n+1$, then *G* is Hamiltionian.

Proof. The statement is true if $ab \in E$, for this we may assume, $ab \notin E$, without loss of generality. assume $d(a,G)\geq d(b,G)$.

Let $P = az_1 \ldots z_{2n-2^b}$ be a Hamiltionian path in *G*. Since $d(a,G)+d(b,G)\geq n+1$, then there is *k* such that $\{az_{2k+1}, bz_{2k}\}\subseteq E$, for some *k*, and $\{a, z_{2k}\}\subseteq A, \{z_{2k+1}, b\}\subseteq B$. Therefore, *G* contains a Hamiltionian cycle, $C = P' \cup P'' \cup \{bz_{2k}, az_{2k+1}\}$ where P'', P' are the (a-z_{2k}) and (z_{2k+1}-b) subpaths of *p* respectively.

Hence, C=az_1...z_{2k} bz_{2n-1}...z_{2k+1}a is a Hamiltionian cycle of *G*, i.e. *G* is Hamiltionian, see sceam.4.2.2.



sceam.4

Lemma:

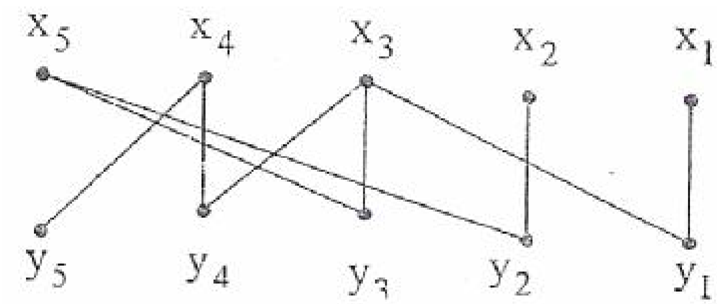
If there is a partition of G into two paths with endvertices (a_1, b_1) and $(a_2, b_2), a_i \in A, b_i \in B$ such that,

$$d(a_1, G) + d(b_2, G) \geq n+1,$$
$$d(a_2, G) + d(b_1, G) \geq n+1,$$

then G is Hamiltionian.

1- Example:

Let G be a bipartite graph as shown in the following diagram:



And $\{(x_1, y_i), i = 24\} \cup \{(x_j, y_3), j = 3,4\}$

$\cup \{(x_2, y_1), (x_2, y_4)\} \cup \{(y_5, x_i), i = 3,5\} \subseteq E$.

Namely, $P_1 = x_1 y_1 x_3 y_4 x_4 y_5$;

$P_2 = x_2 y_2 x_5 y_3, V(P_1) \cup V(P_2) = V(G)$.

And $V(P_i) \cap V(P_2) = \phi$. Hence by Lemma, G contains a Hamiltionian cycle say;

$$x_1 y_1 x_2 y_2 x_5 y_3 x_3 y_5 x_4 y_4 x_1$$

Lemma:

If Γ is a path (a cycle) in G with 2p vertices and if (a, b) is an edge of G with no vertex in Γ, such that $d(a,\Gamma)+d(b,\Gamma)\geq p+1$; then the subgraph $\Gamma+(a,b)$ is traceable (Hamiltionian).

Proof: Assume Γ is cycle with $|\Gamma|=2p$.

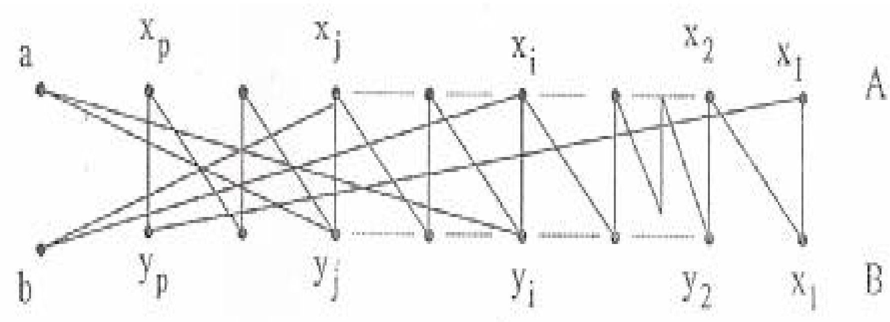
Let $\Gamma=x_1y_1...x_py_p, x_i\in A,\ y_i\in B, i=1,...,p$.

Since $d(a,\Gamma)+d(b,\Gamma)\geq p+1$ and any bipartition. A or B has p points, then there is i, $p-1>i\geq 1$ such that $\{ay_i,bx_i\}\subseteq E$. Hence $\Gamma+(a,b)$ is Hamiltionian i.e. $\Gamma+(a,b)$ has a Hamiltionian cycle, say $x_1y_1...x_ibay_ix_{i+1}y_{i+1}...x_py_p$.

Lemma:

If Γ is a path (a cycle) with 2p vertices and if $a\in A$, $b\in B$ are two vertices not in Γ, such that $d(a,\Gamma)+d(b,\Gamma)\geq p+2$, then the subgraph $\Gamma+(a,b)$ is traceable (Hamiltionian).

Proof: Let Γ be cycle with $|\Gamma|=2p$. Let $\Gamma=x_1y_1...x_py_p, x_i\in A$, $y_i\in B, i=1,2,...,p$. Since $d(a,\Gamma)+d(b,\Gamma)\geq p+2$, then there is some i, j such that $\{ay_i,bx_i,ay_j,bx_j\}\subseteq E$, $\Gamma+(a,b)$ contains a Hamiltionian cycle says; $x_1y_1...x_ibx_jy_{j-1}...y_iay_jx_{j+1}...x_py_px_1$ (see Fig.5).



Sceam.5

Lemma:

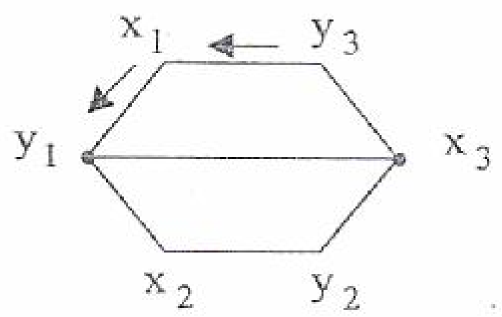
If $a \in A, b \in B$ are vertices of a cycle Γ with $2p$ vertices; such that;

$$d(a^+,\Gamma) + d(b^+,\Gamma) \geq p+2,$$

Γ contains a path P with endvertices a and b, such $V(\Gamma) = V(P)$.

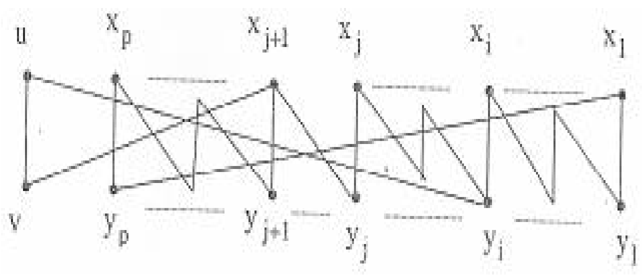
2-Example:

If G be a bipartite graph such that, n=5, Γ is a cycle of length 6, $A = \{x_1,...,x_5\}, B = \{y_1,...,y_6\}, \Gamma = x_1y_1x_2y_2x_3y_3x_1, x_1^+ = y_1, y_2^+ = x_3$
$d(y_1,\Gamma) + d(x_3,\Gamma) \geq 5$, i.e. $y_1x_3 \in E$.



Obviously Γ contains a Path $x_1y_3x_3y_1x_2y_2$.

The proof of Lemma: Let $\Gamma = x_1y_1...x_py_px_1$, with $x_1 \in A$ and $|\Gamma| = 2p$ also, $d(a^+,\Gamma) + d(b^+,\Gamma) \geq p+2$, such that, $a = x_{j+1}, a^+ = y_j$ and $b = y_j, b^+ = x_i$, i.e. $d(y_j,\Gamma) + d(x_i,\Gamma) \geq p+2$. We construct a new graph G' from G by adding two new vertices u and v of degree 2 to G such that y_iuvx_{j+1} is a path of G'. Observe that $p' = (\Gamma - x_2y_i - y_jx_{j+1}) \cup \{y_iu, uv, vx_{j+1}\}$ is a path of G' from x_i to y_j. (see Fig. 4.2.2).



sceam.6

By lemma proved by (Bondy and Charatal; 1976) G' has a cycle C' with $V(C') = C(P')$. Clearly, C' contains the path $y_i uvx_{j+1}$, without loss of generality say $C' = p' x_i$. Hence, the path $p'' = C' - u - v$ statistics the requirement.

Structure Lemma:

Structure lemma: if $n = n_1 + n_2$ and for any, $x \in A, x \in B$, $d(x,G) + d(y,G) \geq n+2$, there is a partition of G into two balanced bipartite subgraphs (G_1, G_2) or (Γ_1, Γ_2) such that one of the following conditions is satisfied:

1. $|V(G_i)| = 2n_i$ and if $x \in A, y \in B$ are in G_i, $d(x,G_i) + d(y,G_i) \geq n_i + 2$.
2. $|\Gamma_1| = 2(n_i - 1), \Gamma_1$ is traceable, $|\Gamma_2| = 2(n_j + 1), j \neq i, \Gamma_2$ is Hamiltionian and if $u \in A, v \in B$ are in Γ_2 $d(u,\Gamma_1) + d(v,\Gamma_2) \geq n_j + 2$,

Proof. Let (G_1,G_2) be a partition of G such that:

(i) $|V(G_i)| = 2n_i$ and G_i contains a Hamiltionian path;

(ii) $e(G_1) + e(G_2)$ is maximum among all partitions that satisfy (i).

If (G_1,G_2) does not satisfy (1), then in one of the subgraphs, say G_1, there are two vertices $a \in A, b \in B$ such that $d(a,G_1) + d(b,G_1) \leq n_1$. Suppose first there is in G_1 a Hamiltionian path with endvertices u and v b. such that $d(u,G_1) + d(v,G_1) \leq n_1$. Then we choose these vertices for a and b. Else, the endvertices, u and v of any Hamiltionian path in G_1 satisfy $d(u,G_1) + d(v,G_1) \geq n_1 + 1$; G_1 is Hamiltionian. Since G_1 is Hamiltionian; then G_1 contains a Hamiltionian cycle say Γ, let a, a^+ and b, b^+ be two endvertices of Γ, so $d(a^+,\Gamma) + d(b^+,\Gamma) \geq p+2$. Thus by Lemma 4.2.6 G contains a Hamiltionian path with endvertices a, b. Hence, G_1 would satisfy the condition (1) of the structure lemma.

Let $\Gamma_1 = G_1 - (a,b); \Gamma_1$ is traceable.

Let $\Gamma_2 = G_2 + (a,b), d(a,\Gamma_2) + d(b,\Gamma_2) \geq b_2 + 2$.

By Lemma Γ_2 is traceable. Suppose there are two vertices $u \in A, v \in B$ in Γ_2 such that;

$$d(u,\Gamma_2) + d(v,\Gamma_2) \leq n_2 + 1$$

A similar argument shows that we can choose u, v such that, $G'_1 = \Gamma_1 + (u,v), G'_2 = \Gamma_2 - (u,v)$ contain Hamiltionian paths on $2n_1$ and $2n_2$ vertices.

But then $e(G'_1) = e(\Gamma_1) + d(u,\Gamma_1) + d(v,\Gamma_1)$

$$e(G'_2) = e(\Gamma_2) + d(u,\Gamma_2) + d(v,\Gamma_2)$$

So $e(G'_1) \geq e(\Gamma_1) + n_1 + 1 \geq e(G'_1) + n_1 + 1 - n_1 = e(G_1) + 1$

And $e(G'_2) \geq e(G_2) + 1$.

Thus $e(G_1) + e(G_2) + 2 \leq e(G'_1) + e(G'_2)$, which contradicts the maximality of $e(G_1)$+$e(G_2)$.

We note $(x_0, y_0, x_1, y_1, ..., x_{n2}, y_{n2})$a Hamiltonian cycle of Γ_2 with $x_i \in A, y_i \in B$. To prove the theorem, we need to know the structure of Γ_2 when Γ_2-(x_0, y_0) is not Hamiltionian.

4.3 Structure of Γ_2 when Γ_2-(x_0, y_0) is not Hamiltonian:

Let $\Gamma'_2 = \Gamma_2 - (x_0, y_0)$. Γ'_2 contains the Hamiltonian path $(x_1 y_1 x_2 y_2 ... x_{n2} y_{n2})$. Γ'_2 is not Hamiltonian. By Lemma and the structure of lemma we obtain;

$$d(x_1, \Gamma'_2) + d(y_{n2}, \Gamma'_2) \leq n_2;$$
$$d(x_1, \Gamma_2) + d(y_{n2}, \Gamma_2) \leq n_2 + 2;$$

Then

x_1 is adjacent to $y_1, y_{j2}, ..., y_{j_p}, 2 \leq j_s \leq n_2 - 1, y_{n_2}$ is adjacent to $x_{i_1}, x_{i_2}, ..., x_{i_{k-1}}, x_{n_2}, 2 \leq i_r \leq n_2 - 1$ with $p + k = n_2$ and $\{1, j_2, ..., j_p\} \cup \{i_1, i_2, ..., i_{k-1}, n_2\} = \{1, 2, ..., n_2\}$.

We have to distinguish two cases.

Case A. there is $k, 2 \le k \le n_2 - 1$, such that the edges (x_1, y_{k+1}) and (x_k, y_{n_2}) exist.

lemma:

If $\Gamma_2 - (x_0, y_0)$ is not Hamiltonian and if the edges (x_1, y_{k+1}) and (x_k, y_{n2}) exist, then x_0 is adjacent to y_k and y_0 is adjacent to x_{k+1} and one of the subgraphs $\Gamma_2 - (x_0, y_2)$ or $\Gamma_2 - (x_1, y_0)$ is Hamiltonian.

Proof. In $\Gamma_2 - (x_0, y_0)$ we consider the Hamiltonian paths:

$$(x_1 y_1 \ldots x_k y_{n2} x_{n2} \ldots x_{k+1} y_k)$$

And $(x_{k+1} y_k \ldots x_1 y_{k+1} \ldots y_{n2})$ (See sceam. 7(a)).

By Lemma 4.2.2 and the structure lemma $d(x_1, \Gamma_2) + d(y_k, \Gamma_2) \ge n_2 + 2$ (note x_1, y_k end vertices of Hamiltonian path of Γ'_2).

$d(x_1, \Gamma'_2) + d(y_k, \Gamma'_2) = n_2$

So $d(y_k, x_0) = 1$.

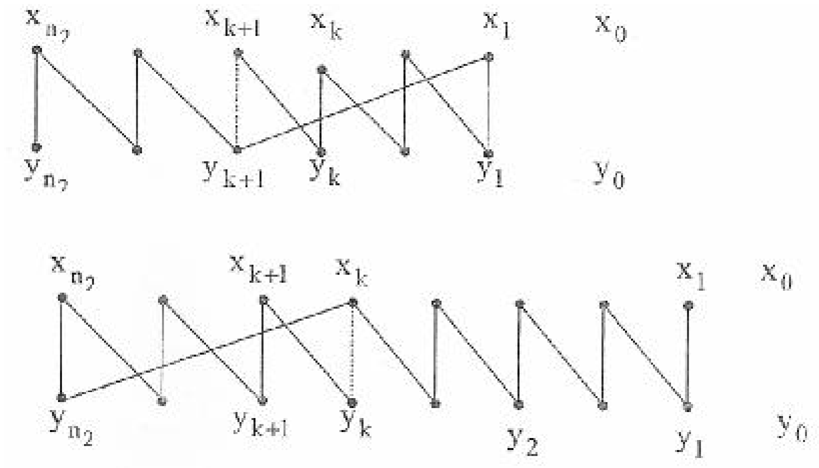
(Γ'_2 is not Hamiltonian, then $d(x_1, \Gamma'_2) + d(y_k, \Gamma'_2) \le n_2$).

A similar argument shows that $d(y_{k+1}, x_0) = 1$. (In this case we take the Hamiltonian path $x_{k+1} - y_{n2}$).

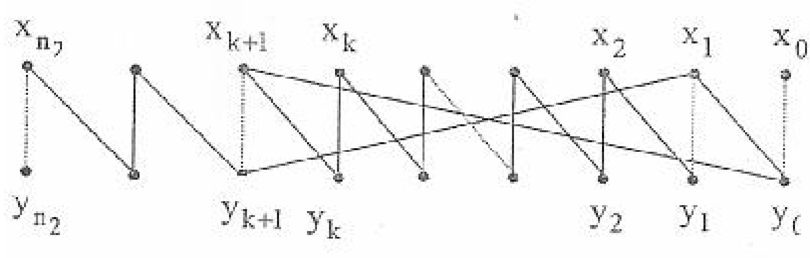
We consider now the Hamiltonian path;

$(y_1 x_2 \ldots x_{k+1} y_0 x_1 y_{k+1} \ldots x_{n_2})$. (See sceam. 8 (b)).

If $\Gamma_2 - (x_0, y_{n2})$ is not Hamiltonian. A similar argument shows that $d(y_1, x_0) = 1$.(In this case we take the $y_1 - x_{n_2}$ Hamiltonian path of $\Gamma_2 - (x_0, y_{n_2})$). Hence $\Gamma_2 - (x_1, y_0)$ is Hamiltonian since $\Gamma_2 - (x_1, y_0)$ contains a Hamiltonian cycle $x_0 y_1 x_2 \ldots x_{n_2} y_{n_2} x_0$. Also if $\Gamma_2 - (x_1, y_0)$ isn't Hamiltonian, by similar argument we conclude the result (take a Hamiltonian path $(y_1 x_2 y_2 \ldots y_{k-1} x_k y_{n_2} x_0 y_k \ldots x_{n_2})$.



sceam. 7 (a)



sceam. 8 (b)

Case B. x_1 is adjacent to $y_1, y_2, ...y_p$ and y_{n_2} is adjacent to $x_{p+1}, ..., x_{n_2}$ for $1 \leq p \leq n_2$.

lemma:

If $\Gamma_2 - (x_0, y_0)$ is not Hamiltonian and if x_1 is adjacent to $y_1, ..., y_p$ and y_{n_2} is adjacent to $x_{p+1}, ..., x_{n_2}$ then

(i) $\{x_1, x_2, ..., x_p, y_{p+1}, y_{p+1}, ..., y_{n_2}\}$, is independent set.

(ii) The subgraphs $(x_1, ..., x_p), (y_0, y_1, ..., y_p)$

$$((y_{p+1}, ..., y_{n_2}), (x_{p+1}, ..., x_{n_2}, x_0))$$

Are complete bipartite subgraphs.

Proof.

(i) Suppose there is an edge (x_i, y_i) for $1 \leq i < p$ and $p+1 \leq j \leq n_2$, then Γ'_2 contains a Hamiltonian cycle $x_1 y_1 ... x_i y_j x_{j+1} ... y_{n_2} x_j y_{j-1} ... y_i x_1$ which is contradict to hypothesis. (See sceam. 9).

(ii) For $1 \le i \le p$ and $p+1 \le j \le n_2$, x_i is independent of $y_{p+1},\ldots,y_{n_2}, y_j$ is independent of $x_1,\ldots,x_p (by(i)); |\Gamma_2| = 2n_2 + 2$,

$$d(x_1,\Gamma_2) \le n_2 + 1 - (n_2 - (p+1) + 1) = p+1 \quad \ldots \quad (1)$$
$$d(y_j,\Gamma_2) \le n_2 + 1 - p \quad \ldots \quad (2)$$

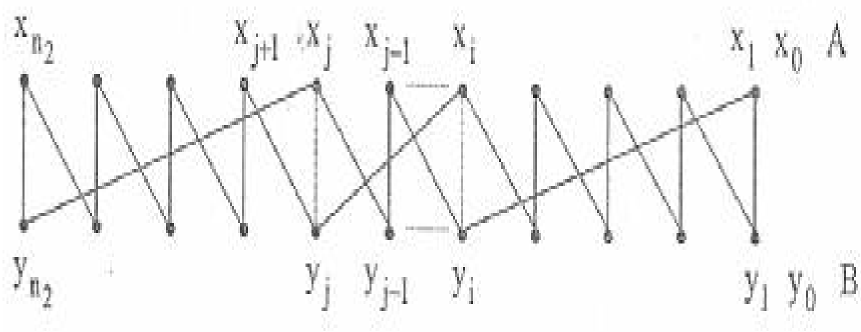
By the structure lemma $d(x_i,\Gamma_2) + d(y_j,\Gamma_2) \le n_2 + 2$. So we obtain equations in (1), (2) and any edge not forbidden exists. Hence,

$$n_2 + 2 \ge d(x_i,\Gamma_2) + d(y_j,\Gamma_2) \ge n_2 + 2$$
$$d(x_i,\Gamma_2) + d(y_j,\Gamma_2) = n_2 + 2$$
$$d(x_i,\Gamma_2) = p+1, d(y_j,\Gamma_2) = n_2 + 1 - p$$

i.e. The subgraphs $((x_1,\ldots,x_p),(y_0,y_p,\ldots,y_p)),$

$$((y_{p+1},y_{p+2},\ldots,y_{n_2}),(x_{p+1},\ldots,x_{n_2},x_p))$$

Are complete bipartite graphs.



sceam.9

Lemma:

In case B, if $\Gamma_2 - (x_1, y_0)$ is not Hamiltonian, if n_2 is odd, Γ_2 is the graph E_1 with $p = (n_2+1)/2$, if n_2 is even, Γ_2 is the graph E_2 with $p=(n_2/2)$. For $1 \le i \le p-1$ and $p+1 \le j \le n_2$, the subgraphs $\Gamma_2 - (x_0, y_j)$ and $\Gamma_2 - (x_1, y_i)$ are Hamiltonian.

Proof: If $\Gamma_2'' = \Gamma_2 - (x_1, y_0)$ is not Hamiltonian:

$d(x_0,\Gamma_2'') + d(y_1,\Gamma_2'') = n_2$, where Γ_2'' has a Hamiltonian path $y_1 x_2 y_2 \ldots y_{n_2} x_0$.

By lemma y_1 is adjacent to $x_2, x_3, \ldots, x_p$ and x_0 is adjacent to $y_{p+1}, \ldots, y_{n_2}$. We distinguish two cases:

Case (i): x_0 is adjacent to y_p and

Case (ii): y_1 is adjacent to x_{p+1}.

Note that case (i) and case (ii) one of there are hold.

By lemma we have in case (i) $\{x_{p+1}, \ldots, x_{n_2}, x_0, y_1, \ldots, y_{p-1}\}$ independent set; and for $1 \le i \le p-1$, $d(x_i, \Gamma_2) = p+1$, $d(y_i, \Gamma_2) = p$, since

x_i is adjacent to $y_1, y_2, \ldots, y_p, y_0$

y_i is adjacent to $x_1, x_2, \ldots, x_p$

for $p+1 \le j \le n_2$, $d(x_j, \Gamma_2) = n_2 + 1 - (p-1) = n_2 - p + 2$; since

x_j is independent of $y_1, \ldots y_{p-1}$

and $d(y_j, \Gamma_2) = n_2 + 1 - p, y_j$ is independent of $x_1, \ldots x_p$

In case ii:

For $1 \le i \le p$, $d(x_i, \Gamma_2) = p+1, d(y_i, \Gamma_2) = p+1$

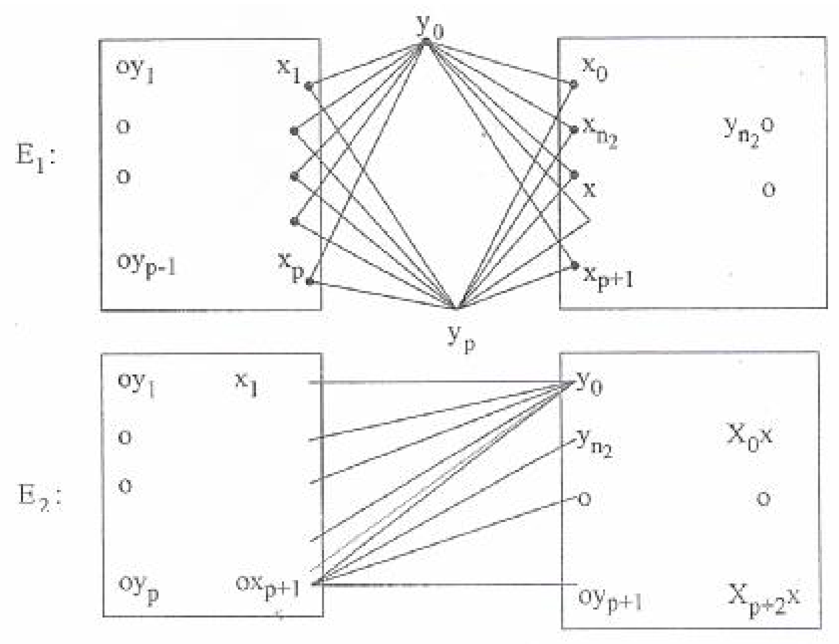
x_i is adjacent to $y_1, \ldots, y_p, y_0$

y_i is adjacent to $x_1, x_2, \ldots, x_p, x_{p+1}$.

For $P+2 \le j \le n_2, d(x_j, \Gamma_2) = n_2 - p + 1, d(y_j, \Gamma_2) = n_2 - p + 1$.

x_i is adjacent to $y_{p+1}, y_{p+2}, y_{n2}, y_0$.

y_i is adjacent to $x_{p+2}, \ldots, x_{n2}, x_0, x_{p+1}$.



sceam.10

By the structure lemma, we obtain

Case (i): $n_2+2=2p+2\left(p=\frac{n_2+1}{2}, n_2 \text{ is odd}\right)$,

Case (ii): $n_2+2=2p+2\left(p=\frac{n_2}{2}, n_2 \text{ is even}\right)$.

If we apply Lemma to $\Gamma_2-(x_0,y_0)$ and $\Gamma_2-(x_1,y_0)$ we obtain respectively:

The graph E_1 if n_2 is odd.

The graph E_2 if n_2 is even. (See sceam. 10).

1- **Example:**

Let $\Gamma_2=x_0y_0x_1y_1x_2y_2x_3y_3x_4y_4$ be a Hamiltonian cycle of G. consider case B as in Lemma 4.8.2, so x_1 is adjacent to y_1,y_2, and y_4 is adjacent to x_3, x_4. Note that $p=\frac{n_2}{2}=2$. As a proof of Lemma in case (ii) y_1 is adjacent to x_3 and we have,

x_0 is adjacent to y_3, y_4.

y_1 is adjacent to x_2.

By Lemma we have in case (ii), $\{x_4, x_0, y_1, y_2\}$ is independent set, if not $\Gamma_2 - (x_1, y_0)$ is a Hamiltionian.

For $1 \le i \le 2$, $d(x_i, \Gamma_2) = 3, d(y_i, \Gamma_2) = 3$

x_i is adjacent to y_1, y_2, y_0

y_i is adjacent to x_1, x_2, x_3.

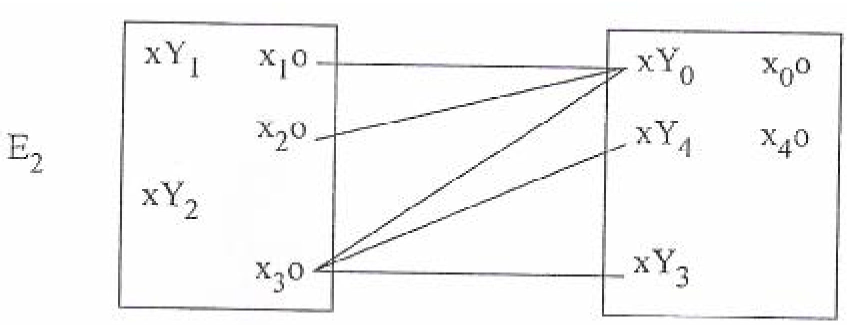
For $j = 4, d(y_4, \Gamma_2) = 3, d(x_4, \Gamma_2) = 3$.

y_4 is adjacent to x_0, x_3, x_4.

x_4 is adjacent to y_0, y_3, y_4.

As the proof of Lemma Γ_2 is the graph E_2 as shown on sceam. 11 Thus $\Gamma_2 - (x_0, y_j)$ and $\Gamma_2 - (x_1, y_i)$ are Hamiltonian for $3 \le j \le 4, i = 1$.

If i=1, j=3, then $\Gamma_2 - (x_0, y_3)$ and $\Gamma_2 - (x_1, y_1)$ contain the Hamiltonian cycles, $x_4 y_0 x_1 y_1 x_2 y_2 x_3 y_4 x_4$ and $x_2 y_2 x_3 y_3 x_4 y_4 x_0 y_0 x_2$ respectively.



Sceam. 11

Proof of the theorem: First Case:

There are two adjacent vertices of Γ_2, adjacent to the endvertices a and b Hamiltonian path of Γ_1.

Let $x \in A, y \in B$ be adjacent vertices of Γ_2 adjacent to b and a. on a cycle of Γ_2 we consider x^+, x^-, y^+, y^-.

If x^+, x^-, y^+, y^- are not adjacent to Γ_1;

$d(x^+,\Gamma_2)+d(y^+,\Gamma_2)\geq n_1+n_2+2,$

$d(x^-,\Gamma_2)+d(y^-,\Gamma_2)\geq n_1+n_2+2$, since we have

$d(x^-,G)+d(y^-,G)\geq n_1+n_2+2\,\&\,d(x^+,G)+d(y^+,G)\geq n_1+n_2+2$.

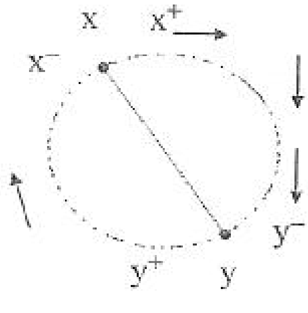
We have Γ_2 in Hamiltonian and $xy\in E$; and

$$d(x^+,\Gamma_2-(x,y))+d(y^+,\Gamma_2-(x,y))\geq n_1+n_2>n_2+1,n_1\geq 2$$
$$d(x^-,\Gamma_2-(x,y))+d(y^-,\Gamma_2-(x,y))\geq n_1+n_2>n_2+1,n_1\geq 2$$

Hence, there is a partition of $\Gamma_2-(x,y)$ into two paths with endvertices (y^+,x^-) and (y^-,x^+). (See sceam. 12). Thus by Lemma, we have $\Gamma_2-(x,y)$ is Hamiltonian, and $\Gamma_1+(x,y)$ and $\Gamma_2-(x,y)$ are solutions of the problem.

Else, let $(x_0y_0x_1y_1...x_{n_2}y_{n_2})$ be a Hamiltonian cycle of Γ_2 such that the chardinality of the pairs of consecutive adjacent to a and b is minimum and suppose (x_0, y_0) be adjacent to b and a. If $\Gamma_2-(x_0,y_0)$ is Hamiltonian, $\Gamma_1+(x_0,y_0)$ and $\Gamma_2-(x_0,y_0)$ are the solutions of the problem.

Suppose $\Gamma_2-(x_0,y_0)$ is not Hamiltonian. We consider case A and Case B the presented paragraph.



sceam.12

Case A:

By Lemma x_0 is adjacent to y_k and y_0 is adjacent to y_{k+1}, we consider the Hamiltonian cycle of $\Gamma_2:(y_0x_1...x_ky_kx_0y_{n_2}...x_{k+1}y_0)$. By hypothesis of minimality, one of edges at least (a, y_k) or (b, x_{k+1}) exists, and $\Gamma_1+(x_0,y_k)$

and $\Gamma_2-(x_0,y_k)$ or $\Gamma_1+(y_0,x_{k+1})$ and $\Gamma_2+(y_0,x_{k+1})$ are the solutions of the problem; since $\Gamma_1+(x_0,y_k)$ has a Hamiltonian cycle say; $y_k a \ldots b x_0 y_k$, also, $\Gamma_2-(x_0,y_k)$ has a Hamiltonian cycles;

$$y_0x_1y_1 \ldots x_ky_{n2}x_{n2} \ldots y_{k+1}x_{k+1}x_0$$

note that in case (x_1,y_{k+1}) & (y_{n2}, x_k) are edges in Γ_2. Similarly; if the other case hold.

Case B. Subcase I:

The endvertices a and b of a Hamiltonian path of Γ_1 are adjacent to three consecutive vertices of Hamiltonian cycle of Γ_2.

We can suppose that a is adjacent to y_0 and b is adjacent to x_0 and x_1.

If $\Gamma_2-(x_1,y_0)$ is a Hamiltionian, $\Gamma_1+(x_1,y_0)$ and $\Gamma_2-(x_1,y_0)$ are solutions of the problem. ($y_0a \ldots bx_1y_0$ is Hamiltonian cycle of $\Gamma_1+(x_1,y_0)$). Else $\Gamma_2-(x_1,y_0)$ is not Hamiltonian, by Lemma, Γ_2 is the graph E_1 if n_2 is odd, the graph E_2 if n_2 is even and the subgraphs $\Gamma_2-(x_0,y_j)$ for $p+1\le j\le n_2$ or $\Gamma_2-(x_1,y_j)$ for $1\le i\le p-1$ are Hamiltonian.

If a is not adjacent to y_i $1\le i\le p-1$, or y_j $p+1\le j\le n_2$ $d(a,\Gamma_2)\le 2$. By lemma, for $p+1\le j\le n_2$ $d(y_j,\Gamma_2)=n_2+1-p$ (Since y_j is independent with vertices $x_1,x_2,\ldots,x_p$, $p+1\le j\le n_2$, then

$d(a,\Gamma_2)+d(y_j,\Gamma_2)\le n_2+3-p;$

$d(a,\Gamma_2)+d(y_j,\Gamma_1)\ge n+2-(n_2+3-p)=n_1+p-1;$

$\Gamma_1+(x_0,y_i)$ contains a Hamiltonian path with endvertices a, y_j say (a, …, b, x_0, y_j). By Lemma 4.2.2, $\Gamma_1+(x_0,y_i)$ is Hamiltonian (since $d(a,\Gamma_1^*)+d(y_j,\Gamma_1^*)\ge n_1+p\ge n_1+1$, where $\Gamma_1^*=\Gamma_1+(x_0,y_i)$).

Hence, $\Gamma_1+(x_0,y_i)$ and $\Gamma_2-(x_0,y_i)$ are solutions of problem.

Subcase II:

The endvertices a and b of a Hamiltonian path of Γ_1 are adjacent to y_k and x_k, $k \neq 0$.

If $1 \leq k \leq p-1$ or $p+2 \leq k \leq n_2$, $\Gamma_1 + (x_k, y_k)$ and $\Gamma_2 - (x_k, y_k)$ are the solution of the problem.

If $k = p$, the edes (x_1, y_p) and (x_0, y_p) exist; the vertices x_0, y_0 and x_p are consecutive on the Hamiltonian cycle $(x_0 y_0 x_p y_{p-1} ... x_1 y_p x_{p+1} ... x_{n_2} y_{n_2} x_0)$ and we can conclude as in the first subcase.

If $k = p+1$, the edes (x_0, y_{p+1}) and (y_{n_2}, x_{p+1}) exist; as a similar argument, the vertices x_0, y_0, y_{p+1} are consecutive on the Hamiltonian cycle $(y_0 x_0 y_{p+1} ... x_{n_2} y_{n_2} x_{p+1} ... x_1)$ and we can conclude as in the first subcase.

Subcase III:

The endvertices a and b of a Hamiltonian path of Γ_1 satisfy $d(a, \Gamma_2) + d(b, \Gamma_2) = n_2 + 2$.

If $1 \leq k \leq p-1$ or $p+1 \leq k \leq n_2 - 1$, $\Gamma_1 + (x_k, y_{k+1})$ and $\Gamma_2 - (y_k, x_{k+1})$ are solution of the problem. Note that $\Gamma_2 - (y_k, x_{k+1})$ has Hamiltonian path with endvertices x_k and y_{k+1}.

If $k = p$, $p \leq n_2 - 2, p+2 \leq j \leq n_2, d(y_j, \Gamma_2) = n_2 - p + 1$ (y_j is adjacent to $(x_{p+1}, x_{p+2}, ..., x_{n_2}, x_0)$; $d(a, \Gamma_2) = p+1$; since a is adjacent to $y_1, \ldots, y_p, y_0$, then

$$d(a, \Gamma_1) + d(y_j, \Gamma_1) \geq d(a, G) + d(y_j, G) - d(a, \Gamma_2) - d(y_j, \Gamma_2)$$
$$\geq n_1 + n_2 + 2 - (n_2 - p + 1) - (p+1);$$

As $d(a, \Gamma_1) + d(y_j, \Gamma_1) \geq n_1$ and $|\Gamma_1| = 2(n_1 - 1)$, then there is an edge uv of Γ_1 with $u \in A$ and $\{av, y_j u\} \subseteq E$; see sceam.12, and we have $bx_j \in E$, for $p+2 \leq j \leq n_2$, so $\Gamma_1 + (x_j, y_j)$ has a Hamiltionian cycle namely

$a,\ldots,uy_jx_jb,\ldots,v,a$. Hence $\Gamma_1+(x_j,y_j)$ is Hamiltionian. Not that $\Gamma_2-(x_j,y_j)$ has a Hamiltionian path with envertices y_{j-1} and x_{j+1} and $\{x_{j+1}y_{n_2},y_{j-1}x_{n_2}\}\subseteq E$.

Thus $\Gamma_1+(x_j,y_j)$ and $\Gamma_2-(x_j,y_j)$ are solution of the problem.

The case $p=n_2-1, p\geq 2$ is similar, and if $p=n_2-1=1$, so $n_2=2$, $p=1$, therefore, a is adjacent to y_1,y_0 and b is adjacent to x_1,x_0, then $d(a,\Gamma_1)+d(b,\Gamma_1)\geq n_1$ so Γ_1 is Hamiltonian by Lemma.

Subcase IV:

For any Hamiltonian path of Γ_1, the endvertices α and β satisfy $d(\alpha,\Gamma_2)+d(\alpha,\Gamma_2)\geq n_2+1$.

Lemma:

Under the hypothesis of subcase IV, Γ_1 is Hamiltonian and if $\alpha\in A$ and $\beta\in B$ are in Γ_1, then

$$d(\alpha,\Gamma_1)+d(\beta,\Gamma_1)\geq n_1+1,$$
$$d(\alpha,\Gamma_2)+d(\beta,\Gamma_2)\leq n_2+1,$$

And there is a Hamiltonian path in Γ_1 with endvertices α and β.

Proof: let α and β be the endvertices of a Hamiltonian path of Γ_1: $d(\alpha,\Gamma_1)+d(\beta,\Gamma_1)\geq n_1+1$.

By Lemma 4.2.2 Γ_1 is Hamiltonian, if α^+ is the successor of α, on a Hamiltonian cycle of $\Gamma_1 : d(\alpha,\Gamma_1)+d(\alpha^+,\Gamma_1)\geq n_1+1$ (α,α^+ are endvertices of a Hamiltonian path of Γ_1).

Suppose that $u\in A$ and $v\in B$ are in Γ_1, and satisfy $d(u,\Gamma_1)+d(v,\Gamma_1)\leq n_1$, then

$$d(u,\Gamma_1)+d(u^+,\Gamma_1)\geq n_1+1;$$

$$d(v,\Gamma_1)+d(v^+,\Gamma_1)\geq n_1+1;$$

Implies that, $d(u^{+},\Gamma_1)+d(v^{+},\Gamma_1)\geq n_1+2$. Therefore by Lemma Γ_1 contains a Hamiltonian path with endvertices *u*, *v*, that contradicts our hypothesis.

Proof of the theorem in subcase IV:

$d(x_1,\Gamma_2)+d(y_{n_2},\Gamma_2)=n_2+2$, so $d(x_1,\Gamma_1)+d(y_{n_2},\Gamma_1)\geq n_1$, thus x_1 and y_{n_2} are adjacent to Γ_1. By lemma 4.8.3 one of the subgraphs $\Gamma_2-(x_1,y_0)$ or $\Gamma_2-(x_0,y_{n_2})$ is Hamiltonian.

If $\Gamma_2-(x_1,y_0)$ is a Hamiltonian. Let $\delta\in\Gamma_1$ be adjacent to x_1. By **Lemma;**

$$d(\delta^{+},\Gamma_1)+d(a^{+},\Gamma_1)\geq n_1+1.$$

By Lemma Γ_1 contains a Hamiltonian path with endvertices δ, a. Hence $\Gamma_1+(x_1,y_0)$ is Hamiltonian, since $(y_0 a,\ldots,\delta x_1 y_0)$ is a Hamiltonian cycle of $\Gamma_1+(x_1,y_0)$. Hence $\Gamma_1+(x_1,y_0)$ and $\Gamma_2-(x_1,y_0)$ are solutions of the problem.

Proof of the theorem: Second Case:

For any Hamiltonian path of Γ_1, its endvertices *a* and *b* are not adjacent to two adjacent vertices of Γ_2.

Lemma:

Under the hypothesis of second case, if $\alpha\in A,\beta\in B$ are in Γ_1.

$$d(\alpha,\Gamma_2)\geq 2,\ d(\beta,\Gamma_2)\geq 2$$
$$d(\alpha,\Gamma_1)\geq 2,\ d(\beta,\Gamma_1)\geq n_1+2,$$
$$d(\alpha,\Gamma_2)+d(\beta,\Gamma_2)\leq n_2$$

Proof: Let $u\in A$ in Γ_2 be not adjacent to Γ_1 and $b\in\beta$ in Γ_1, $d(b,\Gamma_2)\geq n+2-d(u,G)-d(b,\Gamma_1)$, since $d(u,G)+d(b,G)\geq n_1+n_2+2$;

$$d(b,G)=d(b,\Gamma_1)+d(b,\Gamma_2);$$
$$d(u,G)=d(u,\Gamma_2)$$

u is not adjacent to Γ_1; and $d(u,G)\leq n_2+1, d(b,\Gamma_1)\leq n_1-1$.

Therefore, $d(b,\Gamma_2) \geq n_1 + n_2 + 2 - n_2 - 1 - n_1 + 1;$

$d(b,\Gamma_2) \geq 2;$

b is adjacent to vertices $x \in A$ in Γ_2. Then $y = x^+$ is not adjacent to Γ_1 (since any two endvertices of Γ_1 is not adjacent to two adjacent vertices of Γ_2).

By a similar argument if $a \in A$ is in Γ_1, $x^+ \in \Gamma_2$, x^+ is not adjacent to Γ_1, $d(a,\Gamma_2) \geq 2$.

Suppose there are $\alpha \in A$, $B \in \beta$, two vertices of Γ_1 that satisfy

$d(\alpha,\Gamma_1) + d(\beta,\Gamma_1) = n_1 + 1$

So $d(\alpha,\Gamma_2) + d(\beta,\Gamma_2) = n_2 + n_1 + 2 - n_1 - 1 = n_2 + 1$.

Necessarily α and β are adjacent to two adjacent vertices of Γ_2 (Since $|\Gamma_2| = 2(n_2 + 1)$ and $d(\alpha,\Gamma_2) + d(\beta,\Gamma_2) = n_2 + 1$), which contradicts our hypothesis:

i.e., $d(\alpha,\Gamma_1) + d(\beta,\Gamma_1) \geq n_1 + 2$

so $d(\alpha,\Gamma_2) + d(\beta,\Gamma_2) \leq n_2$

proof of the theorem in the second case:

Let $a \in A$ and $b \in \beta$ be two vertices of Γ_1, adjacent to $y \in B$ and $x \in A$ in Γ_2. x and y are adjacent to two vertices, consecutive on a Hamiltonian cycle of Γ_2, $y' \in B$ and $x' \in A$ (note that, $x'y' \in E$ and $d(x_j,\Gamma_2) + d(y_j,\Gamma_2) \geq n_2 + 2$.

Let Γ_1'' is obviously Hamiltonian, since $x'ya,\ldots,bxy'x'$ is Hamiltonian cycle of Γ_1''.

Let $u \in A$ and $v \in B$ be two vertices of Γ_2''. We distinguish three cases:

(i) u and v are not adjacent to Γ_1.

(ii) u is adjacent to Γ_1 and then u is not adjacent to y (since y is adjacent to Γ_1).

(iii) u and v are adjacent to Γ_1 and there are not adjacent to y and x.

In each case we can conclude that:

$$d(u,\Gamma_2'')+d(v,\Gamma_2'')\geq n_2;$$

And by Lemma 4.6.1, Γ_2'' is Hamiltonian $(|\Gamma_2''|=2(n_2-1))$.

Let $(ab\alpha_2\beta_2...\alpha_{n_1-1}\beta_{n_1-1})$ be a Hamiltonian cycle of Γ_1 and; if $n_1\geq 5$, let for $3\leq i\leq n_1-2, \alpha_i=\alpha, \beta_i=\beta$ be two vertices of Γ_1 different from a and b. α and β are adjacent to Γ_2'' in $y_1\in B$ and $x_1\in A(x_1y_1\notin E)$. If x_1^+ and y_1^+ are the successors of x_1 and y_1 on Hamiltonian cycle of Γ_2''.

$$d(y_1^+,\Gamma_2)+d(x_1^+,\Gamma_2)\geq n_1+n_2+2;$$

(since x_1^+, y_1^+ are not adjacent to Γ_1), then

$$d(y_1^+,\Gamma_2'')+d(x_1^+,\Gamma_2'')\geq n_1+n_2-2, |\Gamma_2''|=2(n_2-1)\geq n_2+3$$

By Lemma 4.2.6 Γ_2'' contains a Hamiltonian path with endvertices x_1, y_1 respectively; hence $\Gamma_2''+(\alpha,\beta)$ is Hamiltionian $(\alpha_1 y_1,...,x_1\beta_1\alpha_1)$ is Hamiltonian cycle of $\Gamma_2''+(\alpha,\beta)$.

Let $\alpha^-=\beta_{i-1}, \beta^+=\alpha_{i+1}$. By Lemma 4.10.1.

$d(\alpha^-,\Gamma_1)+d(\beta^+,\Gamma_1)\geq n_1+2$.

We can deduce that $\Gamma_1''-(\alpha,\beta)$ is Hamiltonian to illustrate this;

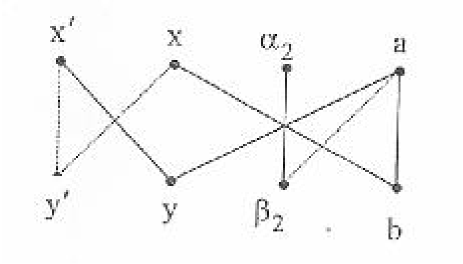
Γ_1'' is Hamiltonian i.e. $x'ya,...,bxy'x'$ is Hamiltonian cycle of Γ_1'', $\alpha=\alpha_i, \beta=\beta_i$ and $\Gamma_1''-(\alpha,\beta)$ is Hamiltonian path with endvertices α^-,β^+ and since $d(\alpha^-,\Gamma_1)+d(\beta^+,\Gamma_1)\geq n_1+2$ so,

$$d(\alpha^-,\Gamma_1''-(\alpha,\beta))+d(\beta^+,\Gamma_1''-(\alpha,\beta))\geq n_1+1, |\Gamma_1''-(\alpha,\beta)|=2n_1$$

By Lemma 4.2.2 $\Gamma_1'' - (\alpha,\beta)$ is Hamiltonian. (This case satisfied if at least one of the edges $\alpha^- y$ or $\beta^+ x$ exist).

If $n_1 \geq 5, \Gamma_1'' - (\alpha,\beta)$ and $\Gamma_2'' - (\alpha,\beta)$ are solutions of the problem.

If $n_1 \leq 4$, it's easy case; to see this argument, if n_1-3, then $(ab\alpha_2\beta_2)$ is Hamiltonian cycle of Γ_1, let α_2, β_2 be adjacent to $x_1 \in A$, $y_1 \in B$ in Γ_2''. Then $\Gamma_2'' + (\alpha_2, \beta_2)$ is Hamiltonian (by the previous argument) and it's obviously $\Gamma_1'' - (\alpha_2, \beta_2)$ is Hamiltonian $(abxy'x'ya)$ is Hamiltonian cycle of $\Gamma_1'' - (\alpha_2, \beta_2)$, (See sceam.13), and if n_1=4, similarly.



sceam.13

This completes the proof of the theorem.

Chapter Five

Cycle and Circumference of Graph

5.1 Number of cycles of certain graph:

In this chapter we will discuss and investigate the nature of cycle of simple graphs, in particular the girth and circumference of graphs, improved and tangible results will be given in this field.

Example: (Pure Graph):

Find the number of cycle of Γ:

Γ:

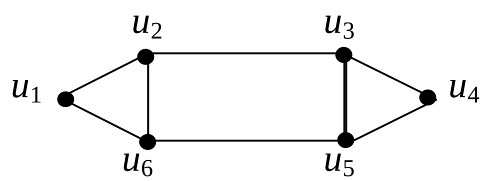


Fig.5.1.1

$C_3 \quad = \{u_1, u_2, u_6, u_1\}$

$\{u_3, u_4, u_5, u_3\}$

$C_4 \quad = \{u_2, u_3, u_5, u_6, u_2\}$

$C_5 \quad = \{u_1, u_2, u_3, u_5, u_6, u_1\}$

$= \{u_2, u_3, u_4, u_5, u_6, u_2\}$

$C_6 \quad = \{u_1, u_2, u_3, u_4, u_5, u_6, u_1\}$

5.2 Girth and Circumference:

Our study will be concerned the girth and circumference of simple connected graphs.

Definition 5.2.1: The girth of a graph G, denoted by g, is the length of the shortest cycle in the graph G.

Example 5.2.1: Figure., shows a 3-regular graph with girth 3.

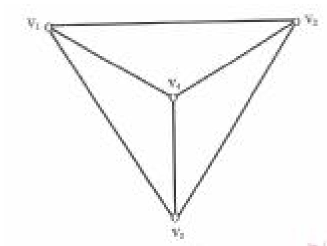


Fig.5.2.1

Definition 5.2.2: Circumference: is the longest cycle in Γ denoted by $C(\Gamma)$.

Example 5.2.2

complete (k_4)- **Graph:**

g (girth) = d+1

= 2+1

= 3

C (circumference) = d+2

= 4

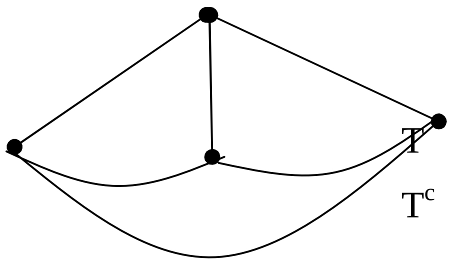


Fig.5.2.2 k_4

Example 5.2.3

Peterson 3-regular

d = 4

g = 4+1 = 5

c=4+2=6

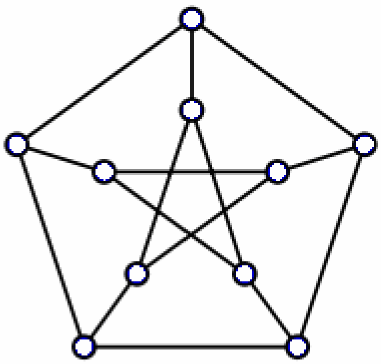


Fig.5.2.3. peterson graph

Theorem 5.2.1: Let Γ be a k-regular graph with diameter d. then $C(\Gamma) = d + (k-1)$.

Proof: Let Γ be a k-regular graph since $\Gamma = T \cup T^c$ (Wrikat 2001).

Let diameter of $\Gamma = d$, i.e. the largest distance between the end vertices in different branches.

Now number of Vertices in the second row is k since Γ is k-regular.

So every branch in the third row has k-1 vertices.

Then the number of branches in the last row is k-1 branches.

Without loss of generality, let x_i in the last row on T^c, so maximum distance from x_i into x_j in another branch is k-1.

Then $C(\Gamma) = d + (k-1)$:

For example k_4, Petersen.

K_4

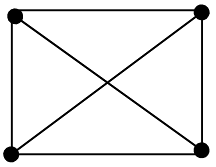


Fig.5.2.4. K_4

d = 4, g = 5, C = 4 + (2) =6.

5.3 Relation between the characteristic polynomial of adjacency matrices and the number of cycle:

Theorem 5.3.1 [Biggs]:

Let $P_\Gamma(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + ... + a_n$ be the characteristic polynomial of adjacency matrices A of Γ. Then $a_i = (-1)^n \sum_X det\, X$, where the sum runs over all $i \times i$ principal submatrices X of A.

Theorem 5.3.2: Let Γ be the wheel graph W(1,n), $P_\Gamma(\lambda) = a_0\lambda^{n-1} + a_1\lambda^n + a_2\lambda^{n-1} + ... + a_{n+1}$ be the characteristic polynomial of adjacency matrices A of Γ, and $n \geq 3$. Then the following hold.

1. $a_0 = (-1)^{n+1}$,
2. $a_1 = 0$,
3. $(-1)^n a_2 = |E(\Gamma)|$,
4. $(-1)^n \dfrac{a_3}{2} =$ the number of k cycles in wheel graph k = 3, 4, …, n-1.

Proof: (1+2) By theorem 4.3.2, for each $i \in \{1,2,\ldots,n+1\}$, the number $(-1)^{n+1} a_i$ is the sum of those principal minors of A which have *i* row and *i* columns. Thus $a_0 = (-1)^{n+1}$ and $a_1 = trace,\ A = 0$.

(3.) A principal minor with two rows and columns, and which has a non zero entry must be of the form $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$. Thus there is one such minor for each pair of adjacent vertices of Γ and each value-1 Hence $(-1)^n a_2 = |E(\Gamma)|$.

(4.) There are essentially three possibilities of non trivial 3×3 principal minors $\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$, and of these the only non-xero is the least where the value is 2 and this principle minor corresponds to three mutually adjacent vertices in Γ some have $a_3 = 2(-1)^n$ (the number of *k* cycle in wheel graph), where *k* = 3, 4, …, *n*-1.

Example 5.3.1:

The adjacency matrix of $W(1,5) =$

$$A(W(1,5)) = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

The characteristic polynomial of $A(W(1,5))$ is

$P_{w(1,5)}(\lambda) = \lambda^6 - 10\lambda^4 - 10\lambda^3 + 10\lambda^2 + 8\lambda - 5$.

The following is special case of proposition 5.5.2.

Theorem 5.3.2. Let Γ be complete graph K_n with,

$$P_{\Gamma}(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots a_n$$

be the characteristic polynomial of adjacency matrices A of Γ. Then the following hold.

1. $a_0 = (-1)^n$,
2. $a_1 = 0$,
3. $(-1)^{n-1} a_2 = |E(\Gamma)|$,
4. $(-1)^{n-1} \dfrac{a_3}{2} =$ the number of *3* cycles in K_n.

Example 5.3.2: The adjacent matrix of K_4=

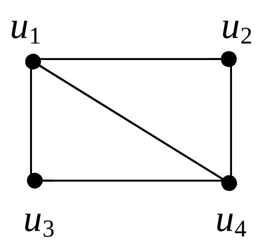


Fig.5.3.1. K_4

$$A = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$, adjacency matrix.

$$|A - \lambda I| = a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_n$$

$$a_n = |A|$$

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$|A-\lambda I|$$

$$A \qquad - \qquad \lambda I$$

$$\left|\begin{bmatrix}0&1&1&1\\1&0&1&0\\1&1&0&1\\1&0&1&0\end{bmatrix}-\begin{bmatrix}\lambda&0&0&0\\0&\lambda&0&0\\0&0&\lambda&0\\0&0&0&\lambda\end{bmatrix}\right|$$

$$=\begin{vmatrix}-\lambda&1&1&1\\1&-\lambda&1&0\\1&1&-\lambda&1\\1&0&1&-\lambda\end{vmatrix}=\lambda^4-3\lambda^2-6\lambda$$

Example 5.3.3: The adjacent matrix of K_5=

$$A(K_5)=\begin{bmatrix}0&1&1&1&1\\1&0&1&1&1\\1&1&0&1&1\\1&1&1&0&1\\1&1&1&1&0\end{bmatrix}$$

The characteristic polynomial of $A(K_5)$ is:

$$P_{K_5}(l)=4+10l\ +20l^2+10l^3-l^5.$$

Theorem 5.3.3. Let Γ be complete bigraph graph $K_{m,n}$ with,

$$P_\Gamma(\lambda)=a_0\lambda^{n+m}+a_1\lambda^{n+m-1}+a_2\lambda^{n+m-2}+\dots a_n \qquad if \quad n>m \quad or$$
$$P_\Gamma(\lambda)=a_0\lambda^{n+m}+a_1\lambda^{n+m-1}+a_2\lambda^{n+m-2}+\dots a_m \qquad if \quad n<m$$

be the characteristic polynomial of adjacency matrices A of Γ. Then the following hold.

1. $a_0=(-1)^{n+m}$,

2. $(-1)^{n+m-1}a_2=|E(\Gamma)|$,

4. $a_1,a_3,a_4,\dots,a_n,a_m=0$.

Remark 5.3.1: The adjacency matrix A of complete bigraph graph $K_{n,m}$ is the form;

$$\begin{bmatrix} 0 & \dots & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & \dots & 1 \\ 1 & \dots & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \dots & 1 & 0 & \dots & 0 \end{bmatrix}$$

Proof of theorem 5.3.4:

Case 1. if $m = n$, then by proposition 3.6.2 we have (1) and (2) since $K_{n,n}$ is *n*-regular graph.

(1.) By theorem 5.3.2 for each $i \in \{1,2,..,n\}$ or $i \in \{1,2,...,m\}$, and $a_i = (-1)^{n+m} \sum_X det\ X$ where *X* is the sum runs over all $i \times i$ principal submatrices *X* of *A*. Thus $a_0 (-1)^{n+m}$.

(2.) A principal minor with two rows and columns, and which has a non zero entry must be of the form $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$. Thus is one such minor for each pair of adjacent vertices of Γ and each value-1 Hence $(-1)^{n+m} a_2 = |E(\Gamma)|$.

(4.) By case (1) part (3) we have (3).

Example 5.3.4: The adjacent matrix of $K_{4,4}$=

$$A(K_{4,4}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The characteristic polynomial of $A(K_{4,4})$ is:

$P_{K_{4,4}}(\lambda) = -16\lambda^6 + \lambda^8$.

And the adjacent matrix of $K_{2,3}$ =

$$A(K_{2,3}) = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

The characteristic polynomial of $A(K_{2,3})$ is:

$$P_{K_{2,3}}(\lambda) = 6\lambda^3 + \lambda^5.$$

Corollary 5.3.1: Let Γ be k-regular graph and let

$P_\Gamma(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_n$ be the characteristic polynomial of the adjacency matrix A of Γ. Then the number of 4-cycle = $\left[a_4 - \frac{1}{2}n[n + (1-2k)]\right]$, where $n = |V(\Gamma)|$.

Proof: By theorem 3.6.3 $\tilde{a}_i = \begin{cases} a_i & for\ \ odd\ i \\ a_i - (-1)^q b_q & for\ i = 2q \end{cases}$.

b_q is the number of basic figures consisting of exactly q non-adjacent edges. When $i = 4, q = 2, b_2 =$ number of pairs of non-adjacent edges which equal to $\frac{1}{2}n[n + (1-2k)]$.

Hence $\tilde{a}_4 = a_4 - b_2$, and since the number of 4-cycles $= \frac{\tilde{a}_4}{2}$, the result holds.

Corollary 5.3.2.The first non vanished item $a_i, i \geq 3$ is the $g(\Gamma) = i$ and number of cycle

$$\begin{cases} \left|\frac{a_i}{2}\right| & if \quad i \;\; is \;\; odd \\ \left|\frac{\tilde{a}_i}{2}\right| & if \quad i \;\; is \;\; even \end{cases}$$

Theorem5.3.4: The largest i of non vanish a_i is the $C(\Gamma)$ if i is odd and $\tilde{a}_i$ if i is even

Corollary5.3.3: $C(\Gamma) \leq |A_n|$ where $\begin{cases} |A_n| & if \;\; i \;\; is \;\; odd \\ |\tilde{A}_n| & if \;\; i \;\; is \;\; even \end{cases}$.

Conclusion:

The cycles of simple Graphs Simple graphs are undirected graphs with no loops and no parallel edges. The matrix theory in general can be considered as a method of using algebra as a tool in graph theory to discuss and study the construction properties of graphs. We study and investigate the cycle nature of well-known graphs and introduce new terms and concepts related to cycles of simple graphs and study and introduce the relation between the operations and the cycle properties of graph such as number of cycles, and cycle lengths. Also we introduce and give matrix representation of certain types of graphs. In particular bigraphs. A new and tangible results on the relation of partition of bipartite Hamiltonian graphs onto two cycles are found, and the relation between the characteristic polynomials of graphs and circumference and girth structure are achieved .

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